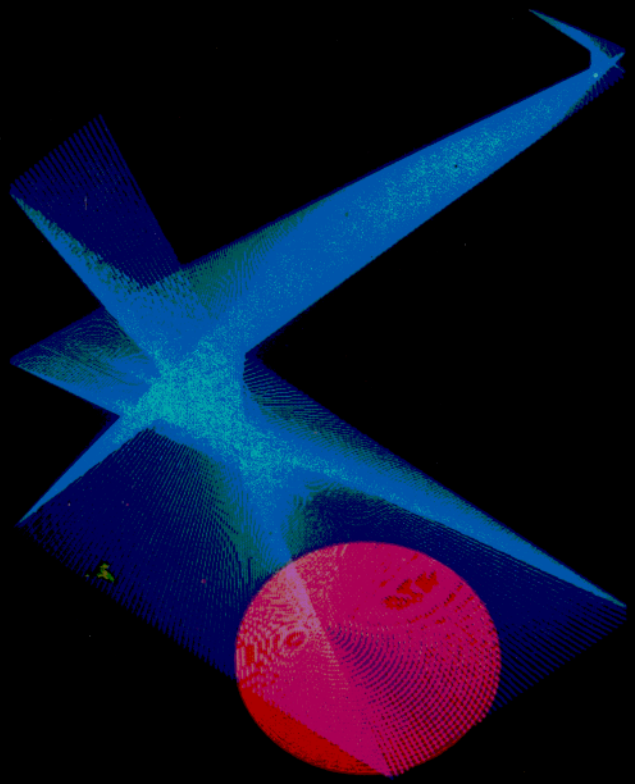
HOW AND WHY IN PHYSICS

2 DYNAMICS



Arvind Kumar ○ Shrish A. Barve

Homi Bhabha Centre for Science Education
Tata Institute of Fundamental Research, Bombay.

Between the formal physics textbooks, problem manuals and general expository books lies the domain of non-formal physics -- the world of content-specific strategies and styles of reasoning that practising physicists employ but do not fully articulate in print. This book tries to capture some elements of this world through the medium of a teacher-student dialogue. The dialogue runs through the many conceptual barriers that most critical students and teachers face in different topics of physics, and offers helpful points, clarifications and insights.

The book should be a valuable accompaniment to, but not a substitute for, the standard introductory text books of physics at senior secondary and introductory college level. It is especially suitable for teachers' orientation courses, and talent-nurture programmes for promising students at that level.

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
Tata Institute of Fundamental Research
V. N. Purao Marg, Mankhurd, Bombay 400 088

PREFACE

The imagined student in this book is concurrently going through a standard introductory course in physics. The student is familiar with most parts of the course and has also practised solving problems. Yet she finds many points unclear and elusive, and she is keen on understanding them critically. The teacher respects the spirit of the student, argues with her on equal terms, and attempts to clarify the points and offer additional insights. The result is a dialogue through different topics of physics, sometimes dwelling on a minor confusion, other times discussing a major conceptual hurdle. Yet the teacher tries to explain with restraint, and judiciously leaves a point when further going might be too technical or advanced.

We believe readers who share the student's doubts on this or that point in introductory physics will find the dialogue useful, and gain some fresh clarifications and insights. But more important, after the book is over, they, like the insatiable student here, should be ready to ask more questions, and yet more questions .....

The work on this series got started mainly due to encouragement by Professors B. M. Udgaonkar and V. G. Kulkarni. Prof. K. Chandrasekharan (formerly of University of Madurai) offered useful comments on the manuscript. Some of the learning barriers in Newtonian Relativity discussed in this book have been investigated in detail in collaboration with our colleagues J. Ramadas and S. Panse. We record our gratitude to all of them. It is a pleasure to thank Shri Anand Ghaisas for help concerning the artwork of the book.

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Here and elsewhere in science, as stressed not least by Henri Poincaré, that view is out of date which used to say, "Define your terms before you proceed". All the laws and theories of physics ... have this deep and subtle character, that they both define the concepts they use ... and make statements about these concepts. ... Any forward step in human knowledge is truly creative in this sense : that theory, concept, law and the method of measurement - forever inseparable - are born into the world in union.

From *GRAVITATION*
(Misner, Thorne and Wheeler)

4 Laws of Motion

Laws and definitions

S1 *Newton's I law states that if there is no force acting on it, a body at rest remains at rest, and a body in uniform motion keeps moving with uniform velocity. Isn't this contrary to experience ? We do seem to require a force to keep a body moving.*

T1 Your observation is right. Indeed the ideas of Aristotle in this regard seem to be more in accord with our ordinary experience. He said that rest is the 'natural' state of a body and every body tends to achieve this state. Motion is unnatural and you need a force to keep a body in motion.

But now we know this to be completely wrong.

S2 *What is wrong with it ?*

T2 You see, in practice, there are always forces which tend to oppose motion (friction, air resistance etc.). If these forces are made smaller and smaller, motion is less and less impeded. So in the ideal case of no friction, uniform motion can continue indefinitely without any external force.

It was Galileo who arrived at this most fundamental insight : no cause is required to keep a body in uniform motion along a straight line (i.e. with constant **v**). Newton's I law is a formal statement of just this insight.

S3 *Is the I law of motion true universally ?*

T3 You must appreciate that the laws of physics relate to physical quantities measured relative to some observer (or frame of reference).[1] To answer your question, the I law of motion is true relative to certain frames of reference (called inertial frames) and not true relative to others (non-inertial frames).

S4 *And how does one decide which frame is inertial and which non-inertial ?*

T4 There is no a priori way of deciding that. The only way to decide whether a frame is inertial or not is to test the validity of Newton's I law. If the law is found empirically to be true, the frame is inertial. If not, the frame is non-inertial.

S5 *This puzzles me. First you say Newton's I law is valid in inertial frames, and then you define an inertial frame to be the one in which the I law is valid.*

T5 Well, it is best to view Newton's I law as a criterion by which one can select, from the set of all frames of reference, a special subset called inertial frames. In other words, Newton's I law only defines an inertial frame.

S6 *But I could use some different criterion for defining some other special class of frames ? What is so significant about inertial frames ?*

T6 Inertial frames are significant because the laws of motion take their simplest form in these frames. Newton's I law, as you will agree, is a rather simple criterion. It says that for the simple situation of no force, motion is also simple (namely uniform motion). Likewise, the II law of motion takes its simplest form in an inertial frame. In non-inertial frames it gets complicated by additional effects as you will see later.

S7 *You said that to see whether a frame is inertial or not, I should check the I law, i.e. test whether motion of a body is uniform in the absence of forces. But what does one mean by a force ? How can I be sure that this as yet undefined thing is zero ?*

T7 You must appreciate that at the very fundamental level physical reasoning is inevitably circular. You are right: the term force is yet to be defined properly, and still we are

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1 The notion of 'frame of reference' is discussed in detail in Chapter 5.

using it in the criterion for an inertial frame. But we do have some intuitive idea about this quantity. Force is some kind of influence on a body. We know by experience that this influence comes either by contact with other bodies or by non-contact effects such as gravitation or electromagnetism.[†] Thus, to ensure that force is absent, we should see that the body is not under any such external influences.

For example, a body far out in space, far from all gravitating objects, may be said to be in a no-force environment. Now if this body has uniform motion relative to a frame, the frame is inertial.

S8 *That looks like an impractical criterion to use. Can't one judge the inertial nature of a frame, say the laboratory frame, by terrestrial experiments where we cannot get rid of gravity ?*

T8 When it is impossible to create a no-force environment, the proper way to ascertain whether a frame is inertial or not, is to apply the II law to the motion of a test particle. If the observed acceleration of the test particle is accountable in terms of the material forces[††] acting on the particle in accordance with the II law, we conclude that the frame is inertial.

S9 *In an inertial frame Newton's II law states that* $\mathbf{F} = m\,\mathbf{a}$. *The only symbol in this equation that I understand clearly is acceleration* $\mathbf{a}$. *The symbol* $\mathbf{F}$ *denoting 'force' has not been defined independently in a quantitative manner, though we do have some intuitive idea about it. In the same way the symbol m denoting 'mass' has not been defined separately. How can an equation containing two undefined quantities become the most basic law of physics ?*

T9 First of all recall what I said in T7. The second law is a classic example of how, at the most fundamental level, definitions, laws and measurements merge into one circular but mutually consistent scheme. Let us understand this better :

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† The fact that forces such as friction, reaction, tension etc. are also electromagnetic in origin at the microscopic level is irrelevant to our discussion. In mechanics, these forces are treated as phenomenological forces different from gravitational, electric and magnetic forces; and this is the practice we shall adopt throughout.

†† The reader should note the terminology that we shall use throughout. We call forces which are attributable to some physical agencies as material forces. (Examples : gravitational force, electric and magnetic forces, contact forces etc.) These are to be distinguished from pseudo-forces (also called inertial forces) which have kinematic origin, as explained in chapter 5.

Start with an intuitive notion of force as some kind of push. Take two bodies 1 and 2, and let body 2 be made of two identical bodies of type 1. Use the same force and measure accelerations of 1 and 2 in turn. (That the force is the same in the two cases is to be judged intuitively.) It turns out that

$$\frac{a_2}{a_1} = \frac{1}{2} .$$

Now, it is reasonable to say that body 2 contains twice as much matter as body 1. If we denote the 'quantity of matter' in a body by m, and call it the mass of the body, then

$$\frac{m_2}{m_1} = 2 .$$

Conclusion : For a fixed force, $m_1 a_1 = m_2 a_2$ = constant.

We now make a jump and take this equation to be true for any pair of bodies. This, then, enables us to have an operational procedure for determining masses of different bodies : keep the force fixed and obtain ratios of masses from the measured values of accelerations. Assigning some arbitrary value of mass to any one body, the masses of all other bodies can be determined.

We thus arrive at the important notion that to each body we can assign a property called mass. It is additive in character (i.e. if two bodies have masses m_1 and m_2, then their composite has a mass $m_1 + m_2$) and it determines the response of the body to a force. The greater the mass, the lesser the acceleration for a given force. Lesser acceleration means lesser change in the state of motion. The property of a body to resist change in its state of motion is called inertia. The constant m appearing in Newton's II law of motion is, therefore, called inertial mass.

S10 *I see the way you arrived at an operational definition of mass, using the intuitive notion of a fixed force. But I must also have a way of obtaining the measure of force* **F** *independently. How, otherwise, will I test the validity of the equation* $\mathbf{F} = m\,\mathbf{a}$ *?*

T10 You must get this right. You see, the II law defines the force **F** to be equal to m **a**. There is no meaning of checking this equation as it is. $\mathbf{F} = m\,\mathbf{a}$ is true by definition.

S11 *That baffles me. How can the most basic law of nature be a mere definition, which, after all, is our construct ? That way, I can concoct my own definitions. For example, I might define force by the relation* $\mathbf{G} = m a^2 \mathbf{a}$. *What makes one definition more correct than the other ?*

T11 Recall our discussion in T23 of chapter 2. No definition is right or wrong. But a definition can be natural, significant and very useful. Or it can be contrived, trite and useless. The Newtonian definition of force has been found to be incomparably useful. In contrast, your definition is not wrong; it is useless.

S12 *On what grounds do you say that ?*

T12 The great significance of the Newtonian definition of force is due to the following reasons :

First, it is the simplest quantitative definition of force that is consistent with the I law. When $\mathbf{F} = 0$, $\mathbf{F} = m \mathbf{a}$ implies $\mathbf{a} = 0$. Your definition of force in S11 is also consistent with the I law, but you will agree that it is not as simple.

Second, the crucial thing about the Newtonian definition is that simple and universal forms for $\mathbf{F}$ exist in nature. You must be familiar, for example, with the inverse square form of the gravitational force between two masses and the electrostatic force between two charges. The equation $\mathbf{F} = m \mathbf{a}$, as it stands, is just a definition; but combined with a definite form for a force, it yields an equation of motion for a particle under the action of the force. This equation is far from a mere definition. It is a law of nature with verifiable consequences. Other definitions of force that one might concoct will not admit of such simple forms in nature.

Lastly, an equally important thing about the Newtonian definition is that it satisfies the III law : the force $\mathbf{F}_{12}$ on particle 1 by particle 2 is equal and opposite to the force $\mathbf{F}_{21}$ on particle 2 by particle 1, i.e. $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The definition $\mathbf{F} = m \mathbf{a}$ combined with the third law leads to the law of conservation of momentum of an isolated system. Once again, your definition of force stands nowhere in comparison. It does not satisfy the III law for $\mathbf{G}$, i.e. $\mathbf{G}_{12} = -\mathbf{G}_{21}$ is <u>not</u> true in nature.

It is these three statements which convert $\mathbf{F} = m \mathbf{a}$ from a mere definition into one of the most basic laws of physics.

To sum up, the greatness of Newton lay in discovering a definition of force which admitted of several simple and universal features. It is these features that underlie the enormous predictive power of Newtonian mechanics.

Newton's laws for extended bodies

S13 *Let me turn to some other queries. Are Newton's laws to be applied to point particles or are they applicable to extended bodies i.e. bodies of finite size also ?*

T13 Strictly, the law $\mathbf{F} = m\,\mathbf{a}$ is to be applied to a point particle. $\mathbf{F}$ here stands for the total force on the particle by the rest of the universe. In the same way, the III law is a statement on the mutual force between two point particles 1 and 2 : $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

S14 *Then, how come we often apply these same laws to extended bodies ?*

T14 An extended body is made up of a number of particles. By applying Newton's laws to the constituent particles we can arrive at the laws for systems of particles. It turns out that, for an extended body (or more generally for a system of particles), we obtain the equation $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{F}$ stands for the net external force on the body, m is the mass of the body and $\mathbf{a}$ is the acceleration of the centre of mass (CM) of the body. To avoid any confusion, it is better to write it as

$$\mathbf{F}^{(ext)} = m\,\mathbf{a}_{CM} \,. \qquad (1)$$

This equation does not follow from the II law alone; we also need to use the III law. Similarly, if $\mathbf{F}_{AB}$ is the force on body A by body B, and $\mathbf{F}_{BA}$ is the force on body B by body A, then it can be seen, using Newton's laws, that $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$. This is the III law for forces between extended bodies. (See Box 4.1.)

S15 *Is the II law $\mathbf{F} = m\,\mathbf{a}$ true universally ?*

T15 If by $\mathbf{F}$ in this equation we mean only the material forces, i.e. forces arising from physical agencies (e.g. gravitational force, electromagnetic force, contact forces like friction etc.), then the II law is true only in an inertial frame. However, the same

equation is applicable even in a non-inertial frame, provided we include in **F** both material forces as well as suitable frame-dependent forces (called pseudo-forces). We shall talk about this matter in Chapter 5.

Box 4.1

Consider a system of N particles. The II law, applied to the i^{th} particle, yields

$$m_i \boldsymbol{a}_i = \boldsymbol{F}_i = \boldsymbol{F}_i^{(ext)} + \sum_{\substack{j=1 \\ j \neq i}}^{N} \boldsymbol{F}_{ij} , \qquad (2)$$

where $\mathbf{F}_{ij}$ is the force on the i^{th} particle by the j^{th} particle, so that the second term on the right hand side is the net force on the i^{th} particle due to all the remaining particles of the system. $\mathbf{F}_i^{(ext)}$ is the net external force on the i^{th} particle. Summing over all the N particles of the system, we get

$$\sum_{i=1}^{N} m_i \boldsymbol{a}_i = \sum_{i=1}^{N} \boldsymbol{F}_i^{(ext)} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \boldsymbol{F}_{ij}$$

The second term on the right hand side contains pairs of terms like $\mathbf{F}_{12}$, $\mathbf{F}_{21}$; $\mathbf{F}_{45}$, $\mathbf{F}_{54}$ etc. Each pair sums to a null force because of the III law. Therefore

$$\sum_{i=1}^{N} m_i \boldsymbol{a}_i = \boldsymbol{F}^{(ext)} , \qquad (3)$$

where $\mathbf{F}^{(ext)}$ is the vector sum of all the external forces on the different particles of the system.

Now, the CM of the system is defined by

$$\boldsymbol{R} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{m} .$$

where m is the total mass of the system. This gives

$$\sum_{i=1}^{N} m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = m \frac{d^2 \boldsymbol{R}}{dt^2}$$

i.e.

$$\sum_{i=1}^{N} m_i \boldsymbol{a}_i = m \boldsymbol{a}_{CM} .$$

With this, Eq.(3) takes the form

$$m \boldsymbol{a}_{CM} = \boldsymbol{F}^{(ext)} .$$

which is Newton's II law for an extended body.

Consider next the mutual force between two extended bodies A & B. The force $\mathbf{F}_{AB}$ on body A by body B is given by

$$\boldsymbol{F}_{AB} = \sum_{a=1}^{N} \sum_{b=1}^{M} \boldsymbol{F}_{ab}$$

where the index a runs over all the N particles which constitute body A, and the index b runs over all the M particles which constitute body B. The force $\mathbf{F}_{BA}$ on body B by body A is given, similarly, by

$$\boldsymbol{F}_{BA} = \sum_{a=1}^{N} \sum_{b=1}^{M} \boldsymbol{F}_{ba}$$

Since the individual forces are equal and opposite,

$$\boldsymbol{F}_{AB} = - \boldsymbol{F}_{BA}$$

which is Newton's III law for extended bodies.

Variable mass systems

S16 *Well, then. In an inertial frame is the relation* $\boldsymbol{F} = m \boldsymbol{a}$ *true in all situations ?*

T16 In non-relativistic mechanics, where we usually deal with well-defined systems with fixed mass, the II law in the above form is always valid. But, then, there are some systems which have a variable mass. For example, a raindrop falling under gravity and collecting mist on its way has a mass that grows with time. Similarly, a rocket that propels forward by ejecting its combustion products has a mass that decreases with time. In these situations the II law must be seen as an instantaneous relation :

$$\boldsymbol{F}(t) = m(t)\,\boldsymbol{a}(t)\ ,$$

where m(t) is the instantaneous mass of the system. Thus, even when the system we are dealing with is a variable mass system, we apply II law to a system of definite mass: the one which has mass m at time t. $\boldsymbol{F}$, therefore, stands for <u>all</u> the forces acting on this system at that instant. In particular, it must include the force exerted by the mass entering or leaving the system. This will be clear from Box 4.2.

Mass, even of a definite system, is not a strict constant. According to Einstein's theory of relativity, the mass of an object increases with its speed. At ordinary speeds the mass variation is negligible, but for speeds comparable to that of light the mass variation is appreciable and modification of Newtonian law of motion is necessary. See chapter 5.

Box 4.2

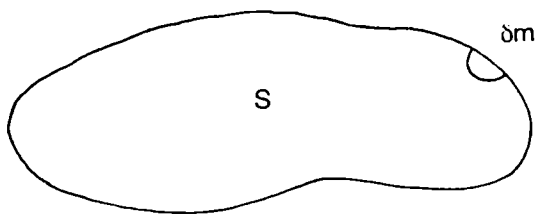

Fig.4.1 Schematic representation of a variable mass system S of mass m at time t. The element of mass δm enters/leaves the system in time δt.

Consider a system with variable mass, shown schematically in Fig.(4.1). At time t, the mass of the system is m. During the small time interval δt, a small mass δm enters or leaves the system so that the mass at time t + δt is m + δm. We apply the II law to the system at time t. Thus the system to which we are applying the II law is S whose mass is m. The net force responsible for the acceleration of the CM of the system S includes, in addition to all the other external forces, the force exerted on S by δm since the mass element δm is external to the system S. Newton's II law gives :

$$m\,\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}_m^{ext} + \boldsymbol{F}_{m,\delta m}\ , \qquad (4)$$

where $\boldsymbol{F}_m^{ext}$ is the net external force on S and $\boldsymbol{F}_{m,\delta m}$ is the force on S by δm. By Newton's III law

$$\boldsymbol{F}_{m,\delta m} = -\,\boldsymbol{F}_{\delta m,m} \qquad (5)$$

Now, if the velocity of δm at time t is $\boldsymbol{v}_b$ and that at t + δt is $\boldsymbol{v}_a$, then the rate of change of

momentum of δm is

$$\left(\frac{dp}{dt}\right)_{\delta m} = \lim_{\delta t \to 0} \frac{|\delta m|\,(v_a - v_b)}{\delta t}.$$

If the change in velocity $(v_a - v_b)$ is infinitesimal, then $(dp/dt)_{\delta m} = 0$. If, however, in the process of entering or leaving the system, the element δm undergoes a finite change in velocity (which is possible if the force on δm is impulsive), then

$$\left(\frac{dp}{dt}\right)_{\delta m} = \left|\frac{dm}{dt}\right| (v_a - v_b) \qquad (6)$$

Now, since $(dp/dt)_{\delta m} = F_{\delta m}^{ext} + F_{\delta m, m}$, we can use Eqs.(5) and (6) to express Eq.(4) in the form

$$m\,\frac{dv}{dt} = F_m^{ext} + F_{\delta m}^{ext} - \left|\frac{dm}{dt}\right| (v_a - v_b)$$

which is the equation of motion for a variable mass system. The force $F_{\delta m}^{ext}$ on the small mass element is usually negligible† and hence we write the equation in the form

$$m\,\frac{dv}{dt} = F_m^{ext} - \left|\frac{dm}{dt}\right| (v_a - v_b)$$

Consider the example of a falling raindrop. The mist droplets floating about in air can be considered to have zero velocity. When a mist droplet sticks to a raindrop, its velocity changes instantaneously to that of the raindrop. Thus, for the mist droplet $v_b = 0$ and $v_a = v$. The equation for this variable mass system can, therefore, be written as

$$\frac{d}{dt}(m\,v) = F^{ext}$$

In case of rocket motion, fuel particles move with the rocket, so that $v_b = v$. After combustion they are ejected with a certain velocity; hence $v_a = v'$. Thus, the equation of motion for a rocket is :

$$m\,\frac{dv}{dt} = F^{ext} - \left|\frac{dm}{dt}\right| (v' - v)$$

Note that $(v' - v)$ is the velocity of the exhaust relative to the rocket. If $F_{ext} = 0$, this equation is easily obtained by momentum conservation as given in many text books.

† In general, however, this is not true. Consider a vertically held chain being dropped on the ground. Every element of the chain, upon hitting the ground, is instantaneously brought to rest by the normal force exerted by the ground. In this case, $(dp/dt)_{\delta m} = F_{\delta m}^{ext}$ and $F_{\delta m, s} = F_{s, \delta m} = 0$.

Newton's III law

S17 *Let us turn to the III law. I have a number of queries about it. First, if the action and reaction forces on a body are equal and opposite, why don't they cancel out and produce a null effect ?*

T17 You have misquoted the III law and, therefore, have arrived at a wrong conclusion. Go back to your words 'action and reaction forces on a body are equal and opposite'. The III law never says that. In the III law we never talk of forces on the same body; rather we are relating force on body A by the body B to the force on body B by the body A. These forces are equal and opposite but they clearly act on different bodies.

Of course, if you consider the composite system (A + B), $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA}$ are internal forces of the system and they do cancel because of the III law. In fact for any system of particles, all the internal forces cancel pairwise because of the III law, as we saw in Box 4.1.

S18 *Can you illustrate the point ?*

T18 A familiar example is collision of two bodies A and B with no external forces. Let us consider body A as our system. It experiences force $\mathbf{F}_{AB}$ during collision. It obviously cannot cancel with $\mathbf{F}_{BA}$ since the latter acts on body B which is not a part of the system under consideration. Consequently, body A suffers a change in momentum. Likewise, if we consider body B as the system to which to apply the Newton's laws, then, an external force $\mathbf{F}_{BA}$ acts on it during collision and the momentum of body B changes accordingly.

We may, however, choose to apply the Newton's laws to the composite system (A + B). $\mathbf{F}_{AB}$ and $\mathbf{F}_{BA}$ are, then, internal forces acting on the same system and it makes sense to add them up. They produce a null result since they are equal and opposite by Newton's III law. There is no other external force on the system, and hence the total momentum of the composite system is conserved.

S19 *Let me ask a standard question. In a horse driven carriage, the horse pulls the cart with some force. By the III law the cart pulls the horse with the same force in the opposite*

direction. For the cart-horse system, therefore, these forces cancel out. What makes the carriage move ?

T19 What you have said is perfectly correct, and if the cart-horse system were far away in empty space, there will be no external force on the system and the momentum of the cart-horse system cannot change.

On the ground, however, the cart-horse system is not isolated. The horse pushes the ground in the backward direction at some angle. By the III law the ground exerts an equal and opposite force on the horse. The vertical component of this force, together with the normal force on the cart, balance the weight of the horse-cart system. The forward component of the force exerted on the horse by the ground causes the motion of the horse-cart system.

S20 *Isn't there a force of friction on the horse ?*

T20 The horizontal force on the horse by the ground in the forward direction is the force of friction between the sole of the horse and the ground. It is what causes the horse to move. In the absence of this friction the horse cannot move.

S21 *Let me turn to another important doubt. In an action-reaction pair, does action precede reaction ?*

T21 No doubt you have got this impression from the terms action and reaction; but it is not true. Remember, no causality whatsoever is implied in Newton's III law. It is wrong to think of action as the cause and reaction as its effect. Both the forces act at the same time, and any of the two can be called action or reaction.

S22 *I am not convinced. For instance, when I press a wall with some force, it is obvious that I am causing the force to act on the wall, and the reaction force by the wall on my hand is an effect or result of what I did.*

T22 You are both right and wrong. You are right because there is indeed a causality in what you have just said. But you are wrong in taking that for a causal relation between the action and reaction forces in the III law.

S23 *Can you please elaborate ?*

T23 For something to be called a cause, it must precede effect. For example, suppose we hear a firecracker explode far away. In this case it is correct to say that the explosion of the firecracker is the cause and the reception of the sound is the effect, because clearly the cause comes before the effect.

But when I press a wall with a certain force, instantaneously an equal and opposite force by the wall acts on my hand; the two forces are simultaneous. So it makes no sense to call one 'the cause' and the other 'the effect'.†

S24 *But I did cause the force to act on the wall, didn't I ?*

T24 True. You decided or intended to press the wall and as a result you did press the wall. There is what is sometimes called an 'intentional causality' here. Physics usually keeps away from analysing intentional causality and the related question of free will etc.

S25 *That is irrelevant to my question. I could replace myself by a robot. The robot could be programmed to undergo a succession of steps, the end result of which is the pressing of the wall by the robot's hand.*

T25 The robot would have to be programmed by a human, so ultimately you cannot get away from the question of intentional causality in this particular example. But suppose you restrict yourself to the robot and the wall. Then there does exist a causality between say the triggering of some switch and the robot hand eventually pressing against the wall. But this, by no means, implies any causal relation between the mutual forces between the robot hand and the wall, which occur simultaneously.

You must realise that the word 'action' in the III law simply stands for a force. It does not have anything to do with the causal succession of steps which lead to that force.

Basically, I think, the wrong notion that action precedes reaction has its origin in the way we use these words in ordinary language. An often misquoted example, albeit in

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† Here we are talking of contact forces. But in Newtonian mechanics even forces such as gravitational force and Coulomb force between bodies at a distance act instantaneously. This, of course, violates special relativity. See chapter 5.

a jocular vain, is : If you hit someone, he hits you back. These two do not form an action-reaction pair in the sense of the III law. When you hit someone, the reaction, in the sense of III law, is what you feel instantaneously in your hand. What the other person delivers later is over and above this reaction force !

S26 *I have heard of weak and strong form of the III law. What is it supposed to mean ?*

T26 In the way we stated the III law, the action and reaction forces are equal and opposite, but we said nothing about their directions relative to the line joining the two particles. This is the weak form of the III law. In the strong form of the III law we say, in addition, that the action and reaction forces are along the line joining the two particles.

S27 *What is the significance of this difference ?*

T27 When the mutual forces between a pair of particles satisfy the strong form of the III law, their lines of action coincide. Clearly, not only the net force but also the net torque on the pair, due to their mutual interaction, is zero. This means, the system by itself can have neither translational acceleration nor rotational acceleration.

On the other hand, if the forces of mutual interaction satisfy only the weak form of the III law, the net force on the system is zero but the net torque need not be zero.

S28 *Can you give some examples ?*

T28 The Newtonian gravitational force between two masses, the electrostatic (Coulomb) force between two charges, the magnetic force between two infinitely long parallel currents are examples of the strong form of the III law.

In contrast, the forces between two magnetic dipoles (or two steady current loops) satisfy only the weak form. Two magnetic dipoles left to themselves can produce a net torque around their centre of mass. See Fig.(4.2).

S29 *Is it correct to say that the III law, at least in the weak form, is valid universally ?*

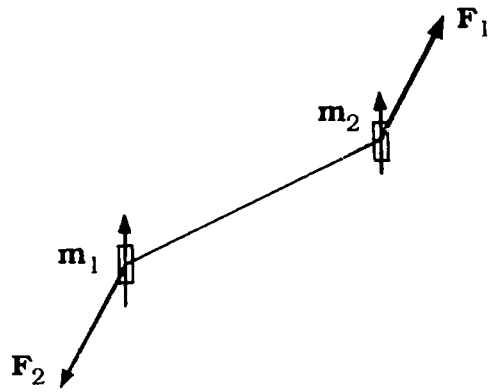

Fig.4.2 The forces of mutual interaction between two magnetic dipoles do not act along the line joining them.

T29 No. The III law is not valid in the relativistic domain i.e. for forces between charged particles in motion. See chapter 5.

S30 *Another question. You said that in a non-inertial frame the II law is modified by the addition of pseudo-forces. Is the III law true in a non-inertial frame ?*

T30 The III law refers to the mutual material forces between a pair of bodies A and B, and says that they are equal and opposite. In effect, it says that material forces in nature always occur in pairs. The III law for material forces is true also in a non-inertial frame. The pseudo-forces in a non-inertial frame, however, do not occur in pairs, and there is no question of a III law for them.

Illustrative problems

S31 *We have discussed so far some general points about the laws of motion. It will help if you take up some practice examples.*

T31 You are right. The best way to understand the laws of motion (indeed any part of physics) is to solve problems. But one must not equate problem-solving to learning

some recipes. We should try to understand critically the principles used while solving a problem.

Let us begin with a very simple example. Consider a box at rest on a table, as shown

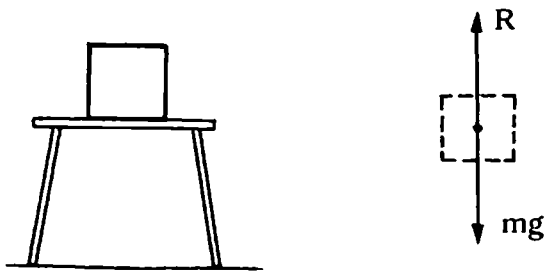

Fig.4.3 A box at rest on a table.

in Fig.(4.3). How do you view the situation in terms of Newton's laws ?

S32 *That is easy. The weight of the box is mg. R is the reaction on the box by the table. Since action and reaction are equal and opposite R = mg, and thus the box is in equilibrium.*

T32 The explanation you have given is completely wrong. The forces R and mg, as shown in the figure, act on the same body. They cannot possibly be an action-reaction pair. The correct action-reaction pairs in the problem are:

gravitational force on the box by the earth = mg (downwards),
gravitational force on the earth by the box = mg (upwards)

and

force on the table by the box = R (downwards),
force on the box by the table = R (upwards)

We see clearly that R and mg do not constitute an action-reaction pair.

S33 *Well, then, how do you explain the equilibrium of the box ?*

T33 It is wrong to think that we can explain the equilibrium of the box. You see, the detailed origin of the force R is so complicated that one cannot determine its value ab initio. All we know about R is that it is a self-adjusting force up to the point the floor does not yield. Thus, if the box is given to be in equilibrium, we use this fact to conclude from the II law (not the III law) that R = mg.

S34 *That surprises me. Does it mean that R may not always be equal and opposite to mg ?*

T34 That is right. It depends on the situation. Suppose, for example, the entire (table + box) assembly were in a lift going downwards with acceleration a. The action-reaction pairs in this case are the same as when the assembly was at rest. In particular, the action force on the table by the box is equal and opposite to the reaction force on the box by the table, since the III law is always true in mechanics no matter what the situation is. But since the box is not in equilibrium, $R \neq mg$. In fact, by the II law,

$$mg - R = ma$$

This again shows how wrong it is to think of R and mg as an action-reaction pair.

S35 *I get your point. Any other example ?*

T35 Consider the situation in Fig.(4.4). Two blocks of masses m_1 and m_2 are connected by an inextensible, massless string. The blocks are being pulled by a third block of mass m_3 via a frictionless pulley as shown in the figure. Determine the acceleration of the blocks.

S36 *This seems straightforward. The force on the blocks m_1 and m_2 is due the weight of the third block i. e. m_3g. By the II law the acceleration is $m_3g/(m_1 + m_2)$.*

T36 Your answer is wrong. You may avoid such errors, while applying the laws of motion, by proceeding systematically step by step :

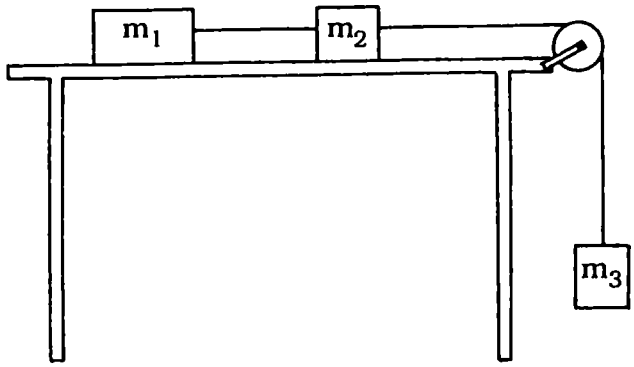

Fig.4.4 Two masses m_1 and m_2 being pulled under the weight of a third mass m_3.

1) Identify clearly the system to which you are applying the laws. The system identified for the application of the II law is usually referred to as 'freebody'. In a given situation, one can take the entire system, or any part of it, as the 'freebody'.[1]

2) Having identified a freebody, it is convenient to represent all the forces acting on the freebody in a diagram. Such a diagram is called a freebody diagram. In this diagram, never include forces by the freebody on other parts of the system.

3) In mechanics, the possible forces on a body are the contact forces such as friction, reaction of a surface or tension in a string etc. and the non-contact force due to gravitation.

4) Sometimes you may need to analyse the situation by taking two or more freebodies one by one. In that case relate appropriate forces using the III law.

S37 *Can you illustrate these points in the problem at hand ?*

T37 Let us first take the mass m_3 as the freebody. The freebody diagram is shown in the Fig.(4.5 a). There are two forces acting on this freebody : the weight m_3g in the downward direction and tension T_2 in the upward direction. If we consider m_2 as the

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1 The term 'freebody' is a misnomer. Avoid a possible wrong impression that 'freebody' means a body on which net force is zero.

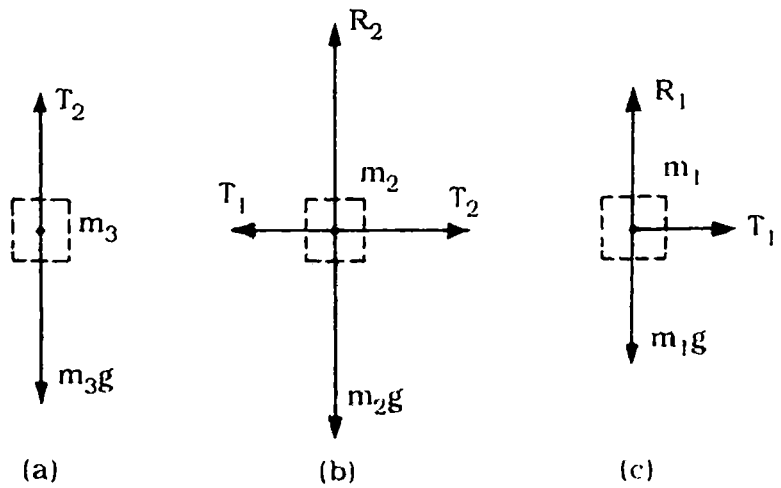

Fig.4.5 Freebody diagrams for the three masses of Fig.(4.4).

freebody, the forces are : weight m_2g downward, the normal reaction R_2 upward, tension T_2 to the right and tension T_1 to the left. Finally, for m_1, the forces are : m_1g downward, the normal reaction R_1 upward and tension T_1 to the right. The freebody diagrams in these two cases are shown in Figs.(4.5 b) and (4.5 c) respectively.

As far as vertical direction is concerned, masses m_1 and m_2 are in equilibrium. Hence $R_1 = m_1g$ and $R_2 = m_2g$. Since the string is inextensible, the accelerations of the three masses must be equal in magnitude. We can therefore write the equations of motion for the three masses :

$$m_1 a = T_1$$
$$m_2 a = T_2 - T_1$$
$$m_3 a = m_3 g - T_2$$

These three equations in the three unknowns a, T_1 and T_2 can be easily solved to obtain $a = m_3g/(m_1 + m_2 + m_3)$.

S38 *One often takes the string to be massless. What if the string has a mass ?*

T38 Consider a string of mass per unit length μ. A small element of length δx is subject to tension T(x) on one side and $T(x + \delta x)$ on the other side, so that the net force on it is $(dT/dx)\delta x$. [See Fig.(4.6)] If a is the acceleration of the string, then

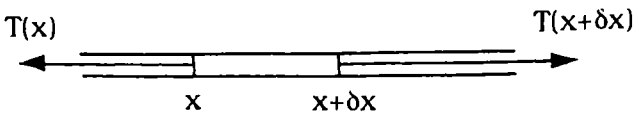

Fig.4.6 Forces on a segment of a string of linear density μ. The tension in the string varies along the length of the string.

$$\mu\, a = \frac{dT}{dx}\,.$$

It is clear that the tension in the string increases linearly along the length of the string. If, however, the string is assumed to be massless, i.e. if $\mu = 0$, then T is constant throughout the string.

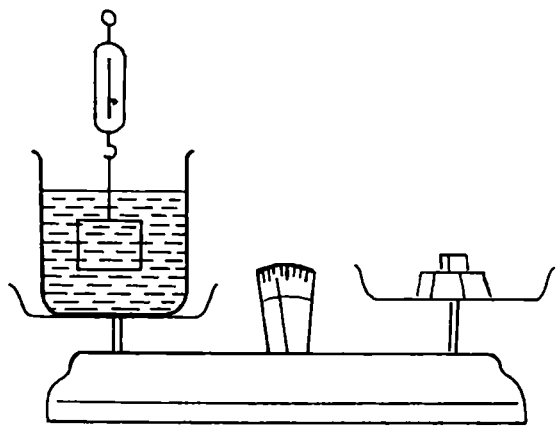

Fig.4.7 The metal block, suspended from a spring balance, is immersed in water in the beaker. The beaker is kept in one pan of a balance.

As next example, consider an arrangement as shown in Fig.(4.7). The weight of the beaker plus water is balanced by 1 kg of mass in the other pan. A piece of a metal of mass 0.08 kg and volume 10^{-5} m^3, suspended by a thread from a spring balance, is immersed in water. How much will the spring balance read ? (Take g = 10 ms^{-2})

S39 *By Archimedes principle, the upthrust on the metal piece, which causes a loss in its*

weight, is equal to the weight of the water displaced. The mass of the water displaced is 0.01 kg. The upthrust, therefore, is 0.1 N and hence the spring balance will read 0.7N, i.e. 0.07 kg.

T39 Fine. Now tell me how much additional mass will I need to put in the other pan for balancing ?

S40 *I do not know how to answer that.*

T40 Choose (beaker + water) as the freebody. Its mass is 1 kg. Now see the extra force on this freebody due to the immersion of the metal piece. Upthrust is the force on the metal piece by the water. By the III law, the metal piece exerts a downthrust on the water of the same magnitude. Thus the extra downward force on the freebody is 0.1N. To balance this, we will need to put an extra 0.01 kg in the other pan.

As our last example, let me take up a very illustrative problem.

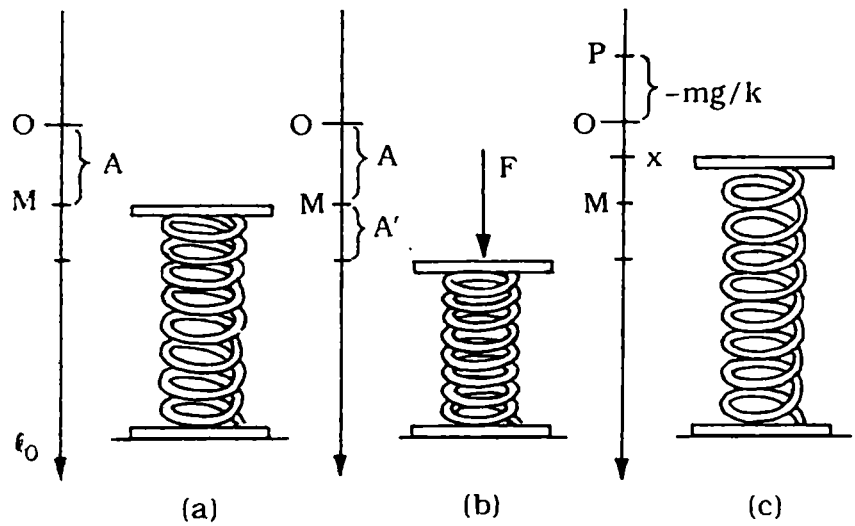

Fig.4.8 The mass-spring system. (a) The spring is compressed under the weight of the upper plate. (b) The force F compresses the spring further. (c) An instantaneous configuration of the assembly after force F is released.

Suppose two plates each of mass m, connected by a massless spring, are kept on a table as shown in Fig.(4.8). The length of the spring in relaxed condition is ℓ_0 and its stiffness constant is k. An external force F compresses the spring. Find the minimum

compression of the spring necessary so that the assembly hops when the force is removed.

S42 *How should I proceed ?*

T42 First consider the assembly kept vertically without any external force as shown in Fig.(4.8 a). Let the compression of the spring in this position be A. How will you draw the freebody diagram for the upper plate ?

S43 *The upper plate has only two forces acting on it : weight mg downward and the spring force – kA (upward). The freebody diagram for the upper plate is as shown in Fig.(4.9a).*

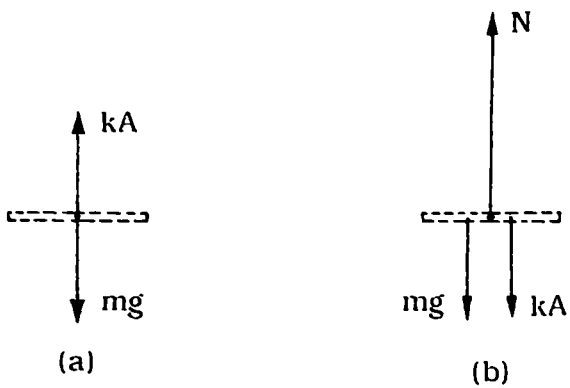

Fig.4.9 The mass-spring system : situation in Fig.(4.8 a). Freebody diagram (a) for the upper plate and (b) for the lower plate.

and the II law gives :

$$mg - kA = 0 \,. \tag{7}$$

T43 Yes: and what does the II law give for the lower plate ?

S44 *Well. the lower plate has its weight mg acting downward. The normal reaction N of the ground should act in the upward direction: and the spring force on the lower plate. I believe. should act in the downward direction.*

T44 That is correct: the spring force is the mutual force between the two plates mediated

by the spring. Hence the spring force on the lower plate must be equal and opposite to that on the upper plate.

S45 *Well then, the freebody diagram for the lower plate is as shown in Fig.(4.9 b) and the II law gives :*

$$mg + kA - N = 0 \qquad (8)$$

T45 Note that we can combine Eqs.(7) and (8) to obtain

$$N = 2m\,g$$

This is just what we should expect, since it describes equilibrium of the system as a whole.

Now suppose an external force F is applied to the upper plate compressing the spring further through a distance A', so that the net compression of the spring is A + A'. (See Fig.4.8 b.) What does the II law yield now ?

S46 *The upper plate now has a force F acting downward in addition to the weight and the spring force. The freebody diagram for the upper plate, therefore, is as shown in*

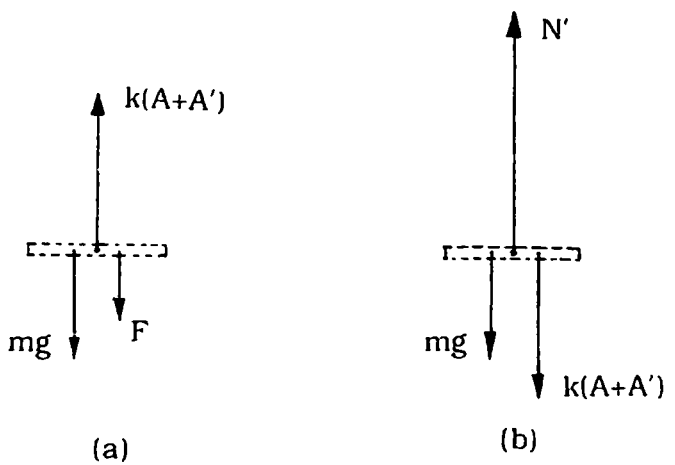

Fig.4.10 The mass-spring system : situation in Fig.4.8 b. Freebody diagram (a) for the upper plate and (b) for the lower plate.

Fig.(4.10 a) and the II law gives :

$$F + mg - k(A + A') = 0 . \qquad (9)$$

For the lower plate, the forces are the same as in the previous case.

T46 Yes, but now the normal reaction must be different.

S47 *Right, so the freebody diagram for the lower plate is as shown in Fig.(4.10 b) and the II law gives*

$$mg + k(A + A') - N' = 0 . \qquad (10)$$

T47 Again notice that, combining Eqs.(9) and (10) we obtain

$$2mg + F - N' = 0 .$$

This equation represents equilibrium of the total system under the new conditions.

Now, when the force F is removed suddenly, the upper plate experiences a net unbalanced force of mg - k (A + A'). It is no longer in equilibrium; its acceleration at that instant is given by

$$mg - k(A + A') = ma .$$

The lower plate, on the other hand, has this self adjusting force of normal reaction acting on it. When the force F is removed, the magnitude of N' reduces instantaneously so that the net force on the lower plate is zero. The upper plate, however, is now in motion. As a result, the compression of the spring, and hence the spring force on the lower plate, varies with time. The normal reaction on the lower plate, therefore, must vary accordingly. As the upper plate moves upward, the compression of the spring decreases, and hence a smaller magnitude of the normal force is enough to keep the lower plate in equilibrium. At the time when the spring is instantaneously in its relaxed position, a non-zero normal reaction must still act on the lower plate to balance its weight. Beyond this point the spring is extended; the spring force on the lower plate is in the upward direction; and the weight of the lower plate is balanced partly by the spring force and partly by the normal reaction. If the extension reaches a point such that the spring force alone is enough to balance the weight of the lower plate, the normal reaction becomes zero. For extension beyond this point, the normal force must act in the downward direction in order to keep the

lower plate in equilibrium, which is impossible. At this point the assembly jumps.

S48 *That seems reasonable, but can we see it in quantitative terms ?*

T48 If we denote the instantaneous compression of the spring by x, then the equilibrium condition for the lower plate yields

$$N(x) = mg + kx ,$$

(See Fig.(4.8 c). Note that distance x is measured from O, taken positive along the downward direction.) It is clear that the normal reaction becomes zero when the compression $x = -mg/k$. (Negative compression implies extension.) For $x > -mg/k$, N(x) is non-negative and the lower plate remains in equilibrium. For $x < -mg/k$, N(x) is negative, which is impossible. Hence at $x = -mg/k$, i.e. when the upper plate reaches point P in Fig.(4.8 c), the assembly jumps.

Recall, however, that the force F compresses the spring through a distance of A'. The upper mass, therefore, performs oscillations with amplitude A' about the mean position M. So if A' is small enough ($A' < A + mg/k$), x will never be less than $-mg/k$, and the assembly will not jump. The minimum compression A' necessary for the spring to jump is, thus, $A + mg/k$.

The force with which the spring must be compressed, in order to produce this compression, is $k(A + mg/k)$. From Eq.(7) we have $kA = mg$. Hence the minimum force with which the spring must be compressed, so that it hops when released, is $2mg$.

S49 *I appreciate the utility of freebody technique for solving problems in mechanics. I suppose I need to solve many more problems to master it better.*

T49 That is right. Let us move to something different.

5 Newtonian Relativity

Frames of reference

S1 *We use the term 'frame of reference' so often in physics. What does it mean precisely?*

T1 Any physical phenomenon is a succession of events. Its description depends on the state of motion of the observer. For example, the position of a box in a moving train remains constant relative to a person in the train but changes in relation to someone on the ground.

For a given state of motion of the observer, the position coordinates of an event depend on the choice of the origin and orientation of the axes. The time of an event depends on the zero setting of the clock that one uses.

To summarize, description of a physical phenomenon is always made relative to some observer in some particular state of motion, with a certain choice of origin, orientation of axes and zero setting of clock.

S2 *I understand that but you still have not explained the term 'frame of reference'.*

T2 Notice that I used words like 'relative to some observer'. The word 'observer', however, can be very misleading. It gives an impression that we are talking of a person looking at the phenomena, making appropriate measurements and possibly comparing them with those of another person. I suggest you banish this picture from your mind.

S3 *I am surprised. What is wrong with it?*

T3 Physics deals with numbers -- measurements carried out by impersonal instruments. The person behind the instruments is irrelevant for physics; that is what one means

by objectivity in science. So it is best to describe phenomena without invoking the notion of an observer, if that word connotes a subjective human observer.

S4 *Then how do you talk of the relativity of description of a phenomenon for different observers ?*

T4 We replace the image of an 'observer' by an impersonal abstract notion of a 'frame of reference'. A 'frame of reference' is no real object; we simply imagine a frame of long rigid rods extending out from a point (origin) in space in three independent directions. This frame of reference is characterized by its state of motion relative to some material body of the universe. Thus, for example, the frame of reference of a train is an abstract artifact which has the same motion as that of the train.

Therefore, instead of saying, for example, that the trajectory of a stone thrown out of a running train is a straight line for a train observer and a parabola for a ground observer, it is better to say that the trajectory is a straight line in the train's frame of reference and a parabola in the ground's frame of reference.

S5 *That sounds like quibbling with words. Isn't that so ?*

T5 In a sense it is. Physicists take the words 'observer' and 'frame of reference' to mean the same thing and often use them interchangeably. You must, however, take care not to have any human-like association with the word 'observer', and understand that 'observations' in physics mean measurements by physical instruments.

A lot of misconceptions on 'frames of reference' arise because one tends to go by the literal meaning of the word 'observer'.

Misconceptions about frames of reference

S6 *Can you warn me of these possible misconceptions about 'frames of reference' ?*

T6 First and foremost, you should never think of a frame of reference as a concrete physical object. A frame of reference is, as said earlier, characterized only by its state

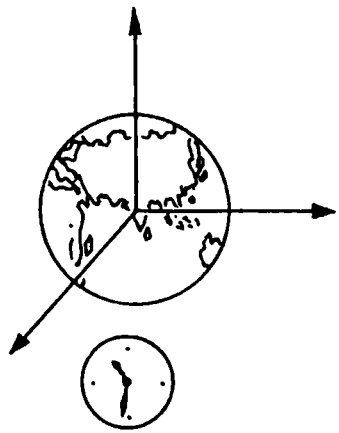

Fig.5.1 A frame of reference of the earth.

of motion. It does not have any other material object-like qualities. For example, when you are talking of a frame of reference of the Earth, the frame is not to be equated to the Earth. The Earth's frame is an abstract artifact whose only relevant characteristic is its state of motion : it has the same motion as the Earth, i.e. relative to this frame the Earth is at rest. In addition, of course, you have to make some choice in this frame of the origin and orientation of the axes, and also of the zero setting of the clock used for measuring time. See Fig.(5.1).

S7 *Can the origin of a train's frame of reference be outside the train?*

T7 Of course. It can be. That is just the pitfall I wanted you to avoid. You must clearly realize that, for the definition of the train's frame of reference, the size and other material characteristics of the train are totally irrelevant. It is wrong to think that the train's frame is 'bounded' by the extension of the train. The origin of this frame can be anywhere in space, and the orientation of its axes can be chosen arbitrarily.

S8 *But in a train's frame of reference, with its origin outside the frame, how can one look at the phenomena inside the train if its windows are shut ?*

T8 Your statement is full of grave misconceptions about 'frames of reference'. You are thinking that there is literally a human observer who positions himself at the origin of the frame and makes observations by 'looking' at the phenomena. I have already said

that in physics observing means measuring: and the process of measurement is not to be equated to viewing by humans. Measurement of any quantity in physics is carried out using some physical instrument. So whatever the origin and the axes of the train's frame of reference, measurements anywhere can be carried out by suitable instruments.

S9 *If a ball is thrown out of a running train, is it right to say that the ball has gone out from the train's frame to the platform's frame of reference ?*

T9 You are making the same error as in S7. As I have already emphasized, a train's frame is not localized or bounded by the train. There is no meaning to the size, territory or boundary of a frame of reference. Every phenomenon can be described relative to any frame of our choice. It is wrong to think that a certain phenomenon 'belongs' to a particular frame. A physical phenomenon stands by itself. Relative to different frames, we have different descriptions of it. Thus the throwing of a ball from a running train to the platform is a phenomenon that can be described relative to both the train's frame and the platform's frame.

As another example, suppose on board a ship at sea, a child is playing with a ring. It is again incorrect to say that this phenomenon 'takes place' in the ship's frame of reference. The phenomenon is apart from the frames. It can be described relative to the ship's frame or the harbour's frame or the frame of reference of a flying aeroplane or any other frame.

In short, we have material bodies in the universe in all kinds of states of motion. With each body we can associate a frame of reference which shares only the motion, but no other characteristic, of the body. Thus we can associate a frame of reference with the sun, the earth, a speeding rocket, an electron in an atom etc., and every phenomenon in the universe can be described relative to any of these frames.

Transformations between frames

S10 *I am now clear about the notion of frames of reference. But suppose I know the description of a phenomenon in one frame, and wish to know what it will be in another frame. what should I do ?*

T10 For that you will need to know how physical quantities like position, velocity, acceleration, etc. transform from one frame to another. That, naturally, depends on what frame you are transforming to. Let us consider various cases :

I. TRANSLATION IN SPACE AND TIME

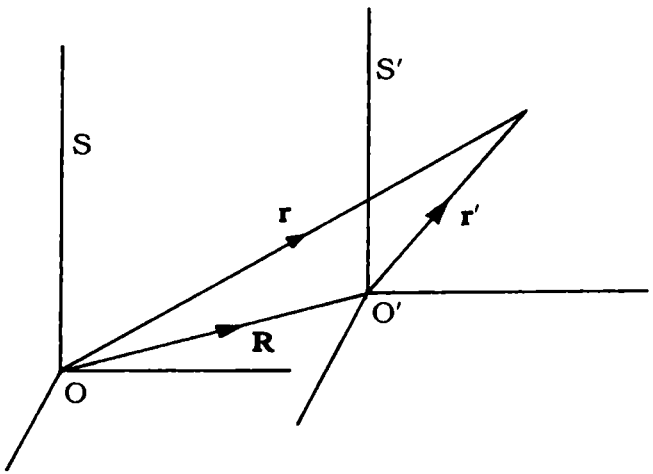

Fig.5.2 The frame S' is displaced relative to S. There is no relative motion between the two frames.

This involves a mere shift of origin as shown in Fig. 5.2; there is no relative motion between S and S'. Clearly, the position vector **r'** of a point relative to S' is related to the position vector **r** of the same point relative to S by the relation

$$\mathbf{r} = \mathbf{r}' + \mathbf{R} \tag{1}$$

where **R** is the position vector of O' relative to O. In component-form this equation reads :†

$$r_i = r'_i + R_i , \qquad i = 1, 2, 3 . \tag{2}$$

We can also include in this case a shift in the zero of the clocks used in S and S'; i.e. the same instant has time coordinates t and t' which differ by a constant,

$$t = t' + T . \tag{3}$$

† We adopt the standard convention that components 1, 2 and 3 refer to the x, y and z components respectively. Thus, $r_1 = y$, $r'_1 = x'$ etc.

S11 *But isn't time an invariant quantity in Newtonian mechanics?*

T11 It is not the absolute value of time, but the time interval between two given events that is invariant in Newtonian theory. Two observers can have different zero settings of their clocks, and therefore can assign different values of times to a given event. For instance, the same instant of time is 5 a.m. in England and 10.30 a.m. in India. However, the duration of say one hour is the same at both the places. This also follows from the above equation, since T is fixed :

$$t_2 - t_1 = t_2' - t_1' \ . \qquad (4)$$

Now, the definitions of velocity and acceleration involve time intervals which, as we have seen, are invariant. Thus, since **R** is a fixed vector, Eqs.(1) and (4) give

$$\boldsymbol{v} = \boldsymbol{v}' \qquad \textit{and} \qquad \boldsymbol{a} = \boldsymbol{a}'$$

which, in terms of components, read

$$v_i = v_i' \ ; \qquad a_i = a_i' \qquad i = 1, 2, 3 \ .$$

S12 *You considered the axes of S' parallel to S. What if the axes are oriented differently ?*

T12 You have anticipated the next case that I am going to consider.

II. TRANSLATION AND ROTATION

In this case the vector relation [Eq.(1)] remains the same. (See Fig.5.3.) The component-form [Eq.(2)], however, modifies to

$$r_i = \sum_{j=1}^{3} U_{ij} \, r_j' + R_i \ . \qquad i = 1, 2, 3$$

where U_{ij} is the transformation matrix for rotation that we discussed in Chapter 2. Likewise, the vector relations for velocity and acceleration are the same as in Case 1, but in terms of components they are :

$$v_i = \sum_{j=1}^{3} U_{ij} \, v_j' \ ; \qquad a_i = \sum_{j=1}^{3} U_{ij} \, a_j' \qquad i = 1, 2, 3 \ .$$

For pure rotations i. e. rotations without any translation, simply put $R_i = 0$ in the transformation relation for co-ordinates above.

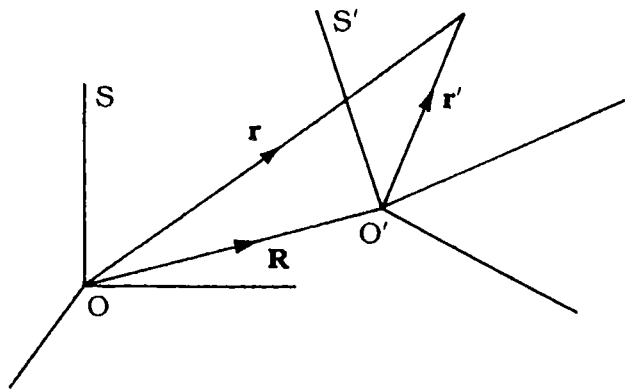

Fig.5.3 The frame S' is displaced and rotated relative to S. The two frames have no relative motion.

Remember, as in case I, there is no relative motion between S and S' here.

S13 *What happens if there is a relative motion between S and S' ?*

T13 This indeed is the situation that most interests us in this Chapter. We shall consider two distinct types of relative motion : relative translational motion and relative rotational motion.

III. RELATIVE TRANSLATIONAL MOTION

Here, S' has pure translational motion relative to S, i.e. O' moves with a velocity **V** with respect to O, but the orientation of the axes of S' remains fixed relative to S. (See Fig.5.4.) Of course, the axes of S' may not be parallel to those of S.

The transformation rule for the position vector is :

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{R} \qquad (5)$$

where, unlike case I, **R** is now a function of time : $d\mathbf{R}/dt = \mathbf{V}$.

The time interval between any two events is always invariant in Newtonian mechanics, no matter what transformations of frames we consider. The transformation rule for time is thus the same as Eq.(3) :

$$t = t' + T \, . \qquad (3)$$

From Eqs.(5) and (3) we get the transformation rules for velocity and acceleration :

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V} \qquad \text{and} \qquad \boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{\alpha} ,$$

where $\boldsymbol{\alpha} = d\boldsymbol{V}/dt$ stands for the acceleration of O′ with respect to O. In component form,

$$v_i = \sum_{j=1}^{3} U_{ij}\, v_j' + V_i \qquad \text{and} \qquad a_i = \sum_{j=1}^{3} U_{ij}\, a_j' + \alpha_i \qquad i = 1, 2, 3 .$$

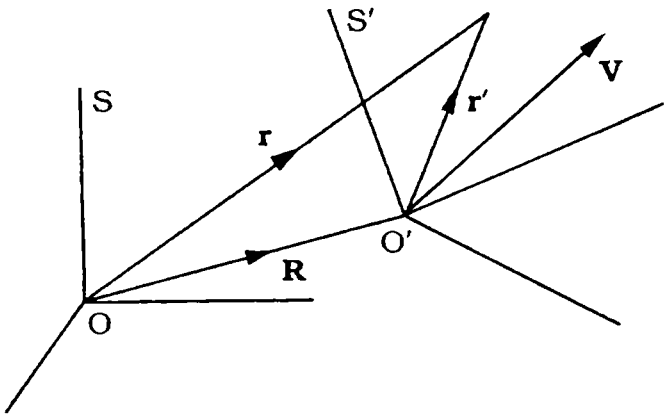

Fig.5.4 The frame S′ moves relative to the frame S with velocity V. The orientation of the axes of S′, however, remains fixed throughout.

The simpler case of uniform relative motion between S and S′ is especially important. Here $\boldsymbol{V}$ is constant so that α is zero. The transformation rules for various quantities in this case become :

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}' + \boldsymbol{R}_0 + \boldsymbol{V}\, t \; ; \qquad t = t' + T \\ \boldsymbol{v} &= \boldsymbol{v}' + \boldsymbol{V} \qquad\quad ; \qquad \boldsymbol{a} = \boldsymbol{a}' \end{aligned} \tag{6}$$

where $\mathbf{R}_0$ gives the location of O′ relative to O at t = 0. Eqs.(6) are sometimes known as Galilean transformations.

For simplicity, one often takes T = 0. Further, for convenience, we take O and O′ to be coincident at t = t′ = 0. The transformation equations for position vector and time then reduce to

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{V}\, t \; ; \qquad t = t' . \tag{7}$$

S14 *I have a query. You said time interval between any two events is invariant in Newtonian theory. Isn't the length or distance also an invariant ?*

T14 You must be careful here. The length of an object say a rod is an invariant. The distance between two events, on the other hand, is <u>not</u> an invariant in general.

S15 *I don't quite get you.*

T15 This seems to be a source of major confusion among students. It will help if you always keep in mind one obvious point. An event is characterized by 'where' and 'when' i.e. its position and time co-ordinates. Now to measure the length of a moving rod, we must measure the position co-ordinates of its two ends simultaneously. Thus length is distance between two <u>simultaneous</u> events.[1] On the other hand the distance travelled by an object, obviously, is the distance between two <u>non-simultaneous</u> events.

From Eq.(6) we have

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}'_2 - \mathbf{r}'_1 + \mathbf{V}(t_2 - t_1) . \tag{8}$$

which clearly shows that the distance between two events at different times is not invariant. Only for simultaneous events is the distance invariant :

$$\mathbf{r}_2 - \mathbf{r}_1 = \mathbf{r}'_2 - \mathbf{r}'_1 \qquad \textit{when} \qquad t_2 = t_1 .$$

Thus the length of an object is an invariant in Newtonian relativity - something that everyone knows intuitively.

S16 *This appears to be an important point. Can you illustrate it ?*

T16 Consider a tube of length 2 m in a moving train as shown in Fig. 5.5. How much do you think is the length of the tube for a ground observer ?

S17 *Obviously 2 m: the length is an invariant quantity.*

---

1 To find the length of a stationary rod, measurements need not be simultaneous. But we are talking of transformation of length from one frame to another. A rod at rest relative to one frame will be moving relative to another frame. So, in general, we must consider simultaneous measurements for determining length.

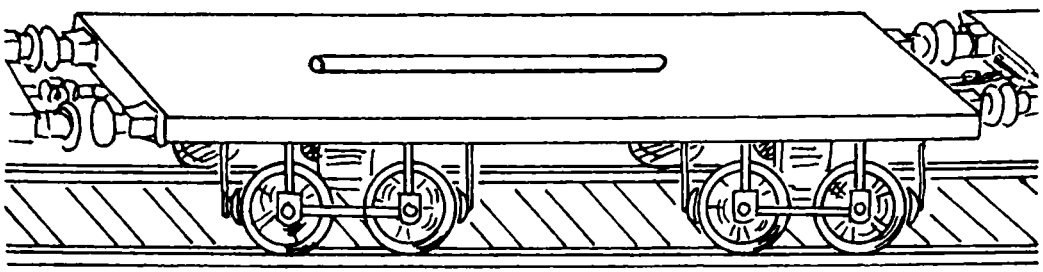

Fig.5.5 A tube on a moving railway carriage.

T17 That is right. Now suppose I roll a ball down from one end of the tube to the other. How much do you think is the distance travelled by the ball relative to the train observer, and the ground observer ?

S18 *Obviously the same, 2 m for both, because the ball has rolled down the tube and the length of the tube is 2 m for both.*

T18 That answer is wrong. This is because the event of the ball starting one end of the tube and the event of its reaching the other end are not simultaneous. The distance between these non-simultaneous events is not the same for the two observers. (See Eq.8.) It is equal to 2 m for the train observer, but different from 2 m for the ground observer. (For example, if the ball moves in the same direction as the train, the distance travelled by the ball for the ground observer is greater than 2 m.)

S19 *How about the time the ball takes to travel from one end of the tube to the other ? Will it be different for the two observers ?*

T19 Of course not. Do not forget that the time interval between any two events is absolute in Newtonian relativity; it is the same for every observer.

S20 *Then how is it that the distance travelled by the ball relative to the two observers is different ?*

T20 Simple. Because the speed of the ball is different for them. Let us see this more explicitly. Suppose the ball moves at a speed of 1 m/s relative to the train. Relative to the train observer, clearly, the distance travelled is 2 m and the time of travel is 2s.

Now suppose the train moves at a speed of 10 m/s. Obviously, the ball moves at a speed of 11 m/s relative to the ground. Since time intervals are invariant, the time of travel relative to the ground observer is the same as that for the train observer, i.e. 2 s. Clearly, therefore, the distance travelled by the ball for the ground observer is 11 m/s × 2 s = 22 m. (If the ball had moved in the direction opposite to the train's motion, the answer would be 9 m/s × 2 s = 18 m)

S21 *I get the point. Any other example ?*

T21 Consider a river flowing with uniform velocity. Suppose there are two trees on the opposite banks of the river such that the line joining them is perpendicular to the flow.

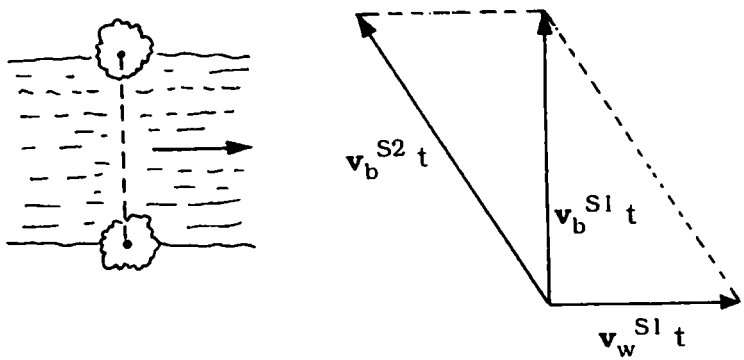

Fig.5.6 The displacement of the boat relative to water ($v_b^{S2}\ t$) and that of water relative to the ground ($v_w^{S1}\ t$) add up to the displacement of the boat relative to the ground ($v_b^{S1}\ t$) in accordance with Eq.(6).

See Fig. 5.6 a. A boat travels along this line from one tree to the other. Now consider two frames of reference : the ground frame (S1) and a frame moving with the river (S2). Is the distance between the two trees same for the two observers ?

S22 *No, because distance is not invariant as you said above.*

T22 You are wrong. Note the trees have no relative motion. So, the distance between them means simply the length of a rod connecting them, or in this example, the width of the river. This, as you know, is invariant.

Now consider the distance travelled by the boat. Is it the same for the two frames S1 and S2 ?

S23 *I guess so. The boat travels the width of the river, which is the same for both S1 and S2.*

T23 You are wrong again. The distance travelled by the boat is the distance between two non-simultaneous events. This distance is different for the two frames in relative motion; the reason being the same as in the previous example. The time of travel for the boat is the same for S1 and S2. The velocity of the boat, however, is different relative to the two frames and, hence, so are the distances. See Fig. 5.6 b.

S24 *I think it is clear now. Let us return to the general discussion about transformation of quantities between frames. We have so far taken the orientation of axes of S and S' to be fixed. What happens if the relative orientation of the axes changes in time ?*

T24 This is an important case that we consider next.

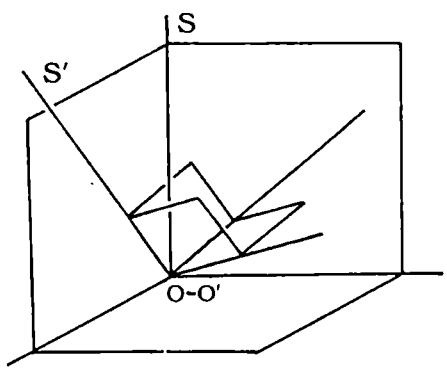

Fig.5.7 The frame S' rotates freely with its origin O' coincident with the origin O of S at all times.

IV. RELATIVE ROTATIONAL MOTION

Here, the axes of S' rotate relative to S with an angular velocity ω while the origins of S and S' remain coincident. (See Fig.5.7.) Evidently, the position vector of any point, $\mathbf{r}$, has different components relative to the two frames. Moreover, the components vary at different rates because of the relative motion between the frames. The relations between velocities and accelerations, as measured relative to the two frames, are derived in Box 5.1. This derivation is a little involved and you may skip it in the first reading. But the results are important for our subsequent discussion. They are as follows :

The relation between the velocity $\mathbf{v} = (d\mathbf{r}/dt)_S$ relative to S and the velocity $\mathbf{v}' = (d\mathbf{r}/dt)_{S'}$ relative to S' is

$$\boldsymbol{v} = \boldsymbol{v}' + \omega \times \boldsymbol{r} \,. \tag{9}$$

The acceleration $\mathbf{a} = (d\mathbf{v}/dt)_S$ relative to S and the acceleration $\mathbf{a}' = (d\mathbf{v}'/dt)_{S'}$ relative to S' are related by the equation

$$\boldsymbol{a} = \boldsymbol{a}' + 2\,\omega \times \boldsymbol{v}' + \omega \times (\omega \times \boldsymbol{r}) + \frac{d\omega}{dt} \times \boldsymbol{r} \,. \tag{10}$$

Box 5.1

A vector which is constant relative to S', obviously, varies with time relative to S. Consider the unit vector $\hat{i}'$ along the x'-axis of the S' frame, which rotates with the frame with angular velocity

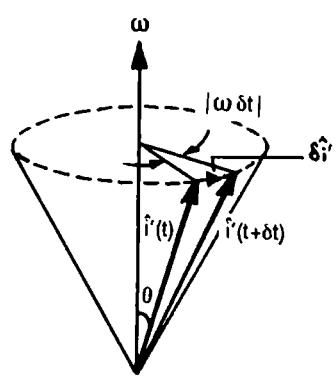

Fig.5.8 Two instantaneous positions of the unit vector $\hat{i}'$ (which is constant in S') relative to S.

ω. Two instantaneous positions of $\hat{i}'$ relative to S, viz. $\hat{i}'(t)$ and $\hat{i}'(t+\delta t)$, are shown in Fig.5.8. It is clear from the figure that

$$|\delta \hat{i}'| = \sin\theta \; |\omega \, \delta t| ,$$

and hence

$$\left| \frac{d\hat{i}'}{dt} \right| = |\omega \times \hat{i}'| .$$

From the figure, the direction of $d\hat{i}'/dt$ matches that of $\omega \times \hat{i}'$. We can thus write

$$\left(\frac{d\hat{i}'}{dt} \right)_S = \omega \times \hat{i}' \tag{11a}$$

Similarly,

$$\left(\frac{d\hat{j}'}{dt} \right)_S = \omega \times \hat{j}' \qquad \text{and} \qquad \left(\frac{d\hat{k}'}{dt} \right)_S = \omega \times \hat{k}' . \tag{11b}$$

The position vector r of any point has different components in S and S' since the two frames differ in the basis vectors :

$$r = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$= x'\hat{i}' + y'\hat{j}' + z'\hat{k}'$$

The rate of change of r relative to S can be calculated using any one of the expansions. If we choose the first one, $\hat{i}$, $\hat{j}$ and $\hat{k}$ are constant vectors, and we get the familiar result :

$$\left(\frac{dr}{dt} \right)_S = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k} .$$

If we choose the second expansion, the fact that $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ are not constant vectors relative to S must be taken into account. Hence

$$\left(\frac{dr}{dt} \right)_S = \frac{dx'}{dt}\,\hat{i}' + \frac{dy'}{dt}\,\hat{j}' + \frac{dz'}{dt}\,\hat{k}' + x' \left(\frac{d\hat{i}'}{dt} \right)_S + y' \left(\frac{d\hat{j}'}{dt} \right)_S + z' \left(\frac{d\hat{k}'}{dt} \right)_S . \tag{12}$$

Since $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ are constant relative to S', clearly,

$$\left(\frac{dr}{dt} \right)_{S'} = \frac{dx'}{dt}\,\hat{i}' + \frac{dy'}{dt}\,\hat{j}' + \frac{dz'}{dt}\,\hat{k}' .$$

Eqs.(11) and (12) then give

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_S = \left(\frac{d\boldsymbol{r}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{r} .$$

which is Eq.(9). Indeed a similar relation is true for the rate of change of any vector **A** relative to the two frames :

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_S = \left(\frac{d\boldsymbol{A}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{A} . \tag{13}$$

In order to find the relation between accelerations relative to the two frames, we differentiate Eq.(9) relative to S :

$$\left(\frac{d\boldsymbol{v}}{dt}\right)_S = \left(\frac{d\boldsymbol{v}'}{dt}\right)_S + \left(\frac{d}{dt}\,\boldsymbol{\omega} \times \boldsymbol{r}\right)_S . \tag{14}$$

For the first term on the right hand side we can use Eq.(13), with **A** replaced by **v'** :

$$\left(\frac{d\boldsymbol{v}'}{dt}\right)_S = \left(\frac{d\boldsymbol{v}'}{dt}\right)_{S'} + \boldsymbol{\omega} \times \boldsymbol{v}' . \tag{15}$$

The second term on the right hand side of Eq.(14) can be evaluated in a straightforward manner:

$$\begin{aligned}\left(\frac{d}{dt}\,\boldsymbol{\omega} \times \boldsymbol{r}\right)_S &= \left(\frac{d\boldsymbol{\omega}}{dt}\right)_S \times \boldsymbol{r} + \boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_S \\ &= \left(\frac{d\boldsymbol{\omega}}{dt}\right)_S \times \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\end{aligned} \tag{16}$$

We have used Eq.(9) while writing the second step. Using Eqs.(15) and (16) in Eq.(14) we obtain

$$\boldsymbol{a} = \boldsymbol{a}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_S \times \boldsymbol{r} ,$$

which is Eq.(10).

Inertial and non-inertial frames / pseudo-forces

S25 *I understand how to make transformations from one frame to another. Let me turn to another theme. Frames of reference are classified as inertial or non-inertial. How to decide which frame is inertial and which non-inertial ?*

T25 The prescription is simple, and it was explained in chapter 4. If, relative to a given frame of reference, the I law is valid then the frame is inertial. If it is difficult to have a no-force situation, we can test the inertial character of a frame by using the second law $\mathbf{F} = m\,\mathbf{a}$, where $\mathbf{F}$ is the resultant of all the material forces acting on the test body. If the equation is satisfied, the frame is inertial; otherwise it is non-inertial.

S26 *Then what is the law of motion in a non-inertial frame ?*

T26 In a non-inertial frame, additional terms need to be put in so that we get a kind of modified form of the II law

$$m\,\mathbf{a} = \mathbf{F} + \mathbf{F}_p \,. \qquad (17)$$

$\mathbf{F}_p$ are called the pseudo-forces which are to be distinguished from material forces such as gravitational, electromagnetic, contact forces etc. They are still qualified as 'forces' because they sit beside the material forces $\mathbf{F}$ in the modified II law, and the two together are responsible for the observed acceleration.

S27 *I am not clear about the nature and origin of pseudo-forces in a non-inertial frame. Please clarify.*

T27 Pseudo-forces are frame-dependent forces. This means the following :

In a given non-inertial frame, if you experiment on different bodies, it is found that, in order to account for the observed acceleration of the bodies, besides the material forces $\mathbf{F}$, you have to introduce certain additional terms $\mathbf{F}_p$. These terms not only depend on the mass, velocity and acceleration of a body, they also involve additional parameters which are common to all the bodies. Now what can these parameters possibly refer to ? We have already said that all the material forces are included in $\mathbf{F}$. So the parameters in $\mathbf{F}_p$ which are common to all bodies cannot possibly refer to any

material agencies. They must refer to the properties of the frame of reference itself.

S28 *I did not quite get you. Can you be more specific?*

T28 You will understand the origin of the pseudo-forces better if you start with the II law of motion in an inertial frame S, and see how it looks like in another frame moving relative to S. This is quite straightforward to do, since you already know how kinematical quantities transform from one frame to another.

1) First consider a frame S′ that is moving with uniform velocity **V** with respect to S. The law of motion in S is :

$$m\,\boldsymbol{a} = \boldsymbol{F}\ . \tag{18}$$

The acceleration $\boldsymbol{a}'$ of a body relative to S′ and acceleration $\boldsymbol{a}$ of the same body relative to S are equal when the relative motion between S and S′ is uniform. In addition, you must realise that the force **F** remains unchanged as we go from S to S′. This is because material forces in mechanics depend only on the (instantaneous) relative positions and velocities of bodies; and it can be seen from Eq.(6) that these quantities are invariant under the transformations under consideration. Further, mass of a body is an invariant quantity in Newtonian mechanics. Consequently, the law of motion in S′ is :

$$m\,\boldsymbol{a}' = \boldsymbol{F}\ .$$

which is identical in form as in S.

This is a very significant thing. It shows that if S is an inertial frame, any frame moving with uniform velocity relative to S is also inertial; i.e. the form of the II law is the same in all inertial frames.

2. Now consider a frame S′ that has arbitrary translational motion relative to S. As seen in T13 the relation between $\boldsymbol{a}$ and $\boldsymbol{a}'$ is

$$\boldsymbol{a} = \boldsymbol{a}' + \boldsymbol{\alpha}\ .$$

where, $\boldsymbol{\alpha}$ is the acceleration of S′ relative to S. Using this relation in the law of motion in S [Eq.(18)] we get

$$m\,\boldsymbol{a}' = \boldsymbol{F} - m\,\boldsymbol{\alpha}\ . \tag{19}$$

So you see, in contrast to the earlier case, the form of the II law in S' is different from that in S.

S' is, thus, not an inertial frame. The additional term, $-m\alpha$, appearing in the law of motion is what is called a pseudo-force. The parameter α characterizes the non-inertial frame S'. The pseudo-force $-m\alpha$ is thus a frame-dependent force.

3. Lastly, consider a frame S' rotating with respect to S with angular velocity ω. The relation between accelerations relative to the two frames is given by Eq.(10). Using this in Eq.(18) we obtain

$$m\,\boldsymbol{a}' = \boldsymbol{F} - 2m\,\boldsymbol{\omega} \times \boldsymbol{v}' - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - m\,\frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r}\,. \qquad (20)$$

Once again we see that the form of the II law is not the same as in the inertial frame S.

Thus the rotating frame S' is not an inertial frame. The additional terms in its law of motion involve m, $\mathbf{v}$, $\mathbf{r}$ of the body. But they also involve a parameter ω which does not depend on the body; it characterizes the frame itself. They are the frame-dependent forces mentioned in T27. The second term on the right hand side of Eq.(20) is called Coriolis force. The third term is centrifugal force which is independent of velocity. The last term involves rate of change of ω. In most applications of interest ω is either constant or changes very slowly. We shall ignore the last term in all our subsequent discussion.

To recapitulate, in an inertial frame $m\,\mathbf{a} = \mathbf{F}$. In a non-inertial frame $m\,\mathbf{a} = \mathbf{F} + \mathbf{F}_p$ where $\mathbf{F}_p$ contains frame-dependent parameters such as α and ω.

S29 *I have a question here. Let me return to the frames S and S' that you considered above in case 2. Now suppose I consider a third frame S'' which has the same acceleration α relative to S as S' has, but may have a different velocity. Is it correct to say that S'' is non-inertial with respect to S, but is inertial with respect to S' since the relative acceleration between S' and S'' is zero ?*

T29 Your question clearly shows that you haven't got the essential point of the preceding discussion. I suggest you go through it again. In particular, do read T25 to T27.

The thing that you must understand is that the inertial or non-inertial character of a

frame of reference is a property of the frame itself, which tells you whether or not measurements relative to it obey the laws of motion in their simple form. It is possible to establish this property of a frame empirically without reference to any other frame, as was explained in T27.

However, as we saw, if a frame S is given to be inertial, any frame moving uniformly relative to S is also inertial; and a frame accelerating or rotating relative to S is non-inertial. This by no means implies that inertial or non-inertial character is a bilateral property of two frames. The statement that a frame is inertial or non-inertial with respect to some other frame is absurd.

Thus, in your example, both S′ and S″ are non-inertial, since they both accelerate relative to S which is given to be inertial.

S30 *That is quite clear now. But what is so significant about all this? I mean what is there to make so much fuss about ?*

T30 The significant thing is that nature admits of a special class of frames in which the laws of motion are especially simple. In other frames the laws get complicated by frame-dependent forces. The famous Newton's pail experiment is a vivid demonstration of just this fact. (See Box 5.2.) In short, *inertial frames are the privileged frames of nature.*

Box 5.2

Newton's Pail Experiment

A pail (bucket) of water is suspended from a ceiling by a rope. The rope is twisted about itself to the maximum and is released. The pail begins to rotate, accelerates to its maximum angular velocity ω (when the rope is fully untwisted), and subsequently deaccelerates to rest as the rope gets twisted in the opposite sense.

Consider two observers S and S′ observing the surface of water in the pail. S is at rest in the laboratory. The frame of reference of S′ rotates with a constant angular velocity ω (the same as the maximum angular velocity of the pail). Let us see what laws the two observers report on the shape of water surface in a pail.

Relative to S :

The surface of water in a stationary pail is a plane; the surface of water in a rotating pail is curved (paraboloid), the curvature increasing with angular speed of the pail.

Relative to S' :

The surface of water in a stationary pail is a paraboloid with a particular value for the curvature; the surface of water in a pail is a plane when it has a paricular angular velocity ($-\omega$).

The experiment clearly shows the inequivalence of the frames of reference S and S'. The laws of physics are evidently simpler in S than in S'. S is an inertial frame and S' is non-inertial.

S31 *Is that not against the principle of relativity which says that no frame is more privileged than the other ?*

T31 That sweeping statement is neither true in Newtonian mechanics nor in Einstein's special relativity.[1] What is true in both is that all inertial frames are equivalent. Non-inertial frames are not equivalent to inertial frames from the viewpoint of the laws of physics as we saw in T28.

S32 *What precisely is meant by the statement that two frames are equivalent ?*

T32 Two frames are said to be equivalent if the laws of physics have the same form in both. Beware, this does not mean that all the physical quantities have the same values in both. For example, for two inertial frames in relative motion, the position vector **r** and velocity **v** of a body relative to S will not be the same as **r'** and **v'** of the same body relative to S'. In general, in equivalent frames, it is the relations between the quantities which are the same, not the values of the quantities themselves.

The phrase 'no inertial frame is more privileged than others' is just a rewording of the statement that 'all inertial frames are equivalent'.

S33 *But in actual practice, we do prefer frames. For example, if I have to describe the motion of a ball thrown up by a child in a uniformly moving train, it is better to use the*

1 It is true in Einstein's general relativity in a sense which cannot be explained here.

train's frame than the ground's frame. In the former, the trajectory of the ball is a straight line, while in the latter it is a parabola. Isn't the train's frame more privileged than the ground's frame in this case?

T33 This is a very natural pitfall. Your error lies in equating a convenient frame for a given phenomenon to a privileged frame from the point of view of laws of physics. We have already stressed that equivalence of frames does not mean equality of measured values of quantities. In other words, description of a phenomenon may be different in two frames, even though the frames are equivalent. And one description could well be simpler than the other.

To put it more rigorously, the trajectory of a particle is determined not only by the laws of motion but also by its initial conditions. The initial conditions for a particular problem may be simpler in one frame than in another even if the laws are identical. In your example, the horizontal velocity of the ball has a simple value (namely zero!) in the train's frame but is non-zero in the ground's frame. That is why, even though the laws of motion are identical on the ground and in the uniformly moving train, the trajectory of the ball is different.

In short, remember not to confuse equivalence of frames with identity of descriptions of physical phenomena.

Uniform motion is relative; acceleration/rotation are absolute.

S34 *There is a standard phrase: 'uniform motion is relative' that one comes across so often. Can you clarify just what this phrase means ?*

T34 The basic relevant facts are already known to you. Now physicists are not always dry people. They like to put an important idea across in different catchy ways, partly to convey the essence of the idea, and partly with a view to communicating technical points in a non-technical language.

Let us see what one means by what has become the standard cliche of relativity : uniform motion is relative.

Suppose I am in an inertial frame S and you are in a frame S' moving uniformly relative

to me. Now suppose I am both ignorant and arrogant, and claim that I am 'really' at rest and it is you who is really moving. I claim that my motion that you observe is apparent, not real. You, on the other hand, are also adamant. You insist that it is you who is really at rest, and that I am actually moving. How to solve this deadlock? We approach a physicist.

The physicist's reply will be something like this :

"Each of you can think himself to be at rest and the other to be moving, if that satisfies your petty ego. But physics-wise I have no way to decide in anyone's favour. Relative to S, S' is moving. Relative to S', S is moving in the opposite direction. Both statements are correct. Since the laws of physics are identical in S and S', there is no way in physics to test whether one of them is 'actually' at rest. The notion of absolute rest, therefore, is inadmissible in physics".

After explaining thus, the physicist summarizes the para with a catchy phrase : 'uniform motion is relative', and calls it the fundamental principle of relativity.

35 *But still I can't help feeling that some motions are real and some apparent. For example, in a train's frame, the motion of the ground and everything else does seem to be apparent. On the other hand, the motion of the train relative to the ground seems real. Why does that feeling arise even when I know the principle of relativity?*

35 That feeling is perhaps the result of a number of reasons. First, the Earth's frame is our customary frame of reference. Further, the Earth is in free fall under the Sun's gravity; we do not need to expend any energy to keep it moving. On the hand, we are aware that to have a train start from rest and to keep it in uniform motion (against friction) we need to expend energy. There is also an asymmetry in the situation in regard to the size; the train is much smaller than the surroundings, and it is more natural to concede 'reality' to the motion of the train than to that of the surroundings. Another reason is that initially the train needs to be accelerated, and its subsequent motion is rarely, if ever, strictly uniform.

In this connection let me bring a common experience to your notice. From a train that is just beginning to move, when its motion is smooth, if you look out of a window at another train on an adjacent track, it is difficult to judge if it is your train that is moving or the other. This happens because the various reasons suggested in the para above are not relevant to this experience.

But whatever the cognitive reasons for the feeling regarding real and apparent motion, be clear about one thing. As long as the train is in uniform motion, there is no way in physics to say that the motion of the train relative to the ground is real and that of the ground and surroundings relative to the train is apparent.

S36 *Why do you insist on the word 'uniform' in your statement of the principle of relativity? Why can't we say that motion, in general, is relative ?*

T36 You already know the answer if you care to go back to T28. We know that when a frame S' accelerates or rotates with respect to an inertial frame S, S' is not inertial. The laws in S and S' are then no longer identical. Unlike uniform motion, acceleration and rotation are absolute concepts.

S37 *I didn't get you.*

T37 Go back to our quarrel in T34 with a slight modification. I continue to be in the inertial frame S but now your frame S' is accelerating relative to me. I claim that I am unaccelerated and that it is you who is actually accelerating. You, adamant as before, insist that you are unaccelerated and that I am actually accelerating in the opposite direction. This time when we approach a physicist, he will be on my side.

S38 *Why ? Isn't it true that if relative to S, S' has an acceleration a, then relative to S', the acceleration of S is - a ?*

T38 That is quite correct kinematically. There is no quarrel on what each of us observes about the other. The issue is whether there is a way in physics to judge if a frame of reference is <u>actually</u> accelerated or not?

S39 *How will the physicist decide the issue ?*

T39 The physicist's reply will be something like this:

"This time, unlike the earlier quarrel, I have a way of settling the issue. The law of motion is different in S and S'. The II law in S' [Eq.(19)] involves a frame-dependent parameter α that stands for the acceleration of S' relative to any inertial frame. In S

the law of motion has no such frame-dependent parameter. So an inertial frame like S should be considered unaccelerated. Acceleration of a frame is thus an absolute concept, i.e. in physics there is a way of telling that a particular frame of reference is or is not actually accelerated.

An exactly similar argument goes through if a frame S′ is rotating relative to an inertial frame S. The II law in S′ [Eq.(20)] involves a frame-dependent parameter ω which stands for the angular velocity of the frame relative to any inertial frame. In S the law of motion has no such frame-dependent parameter. So any inertial frame of reference like S should be considered non-rotating, and it is S′ that is really rotating with angular velocity ω. Rotation is thus an absolute concept."

S40 *Can you recapitulate the various points in the discussion so far ?*

T40 If a frame is inertial, so is any other frame moving uniformly relative to it. Inertial frames are equivalent to one another, i.e. they have the same form of the II law. This prevents us from singling out, among these, any one frame which can be claimed to be really at rest.

But inertial frames are not equivalent to non-inertial frames. The presence of frame-dependent forces (pseudo-forces) in the II law for non-inertial frames allows one to claim that these are actually accelerating or rotating, while the inertial frames are unaccelerated and non-rotating.

Perhaps the best way of saying all this is that, by experiments described relative to a given frame of reference, it is possible to determine the intrinsic acceleration / rotation parameters of the frame, but not its velocity. If the acceleration / rotation parameters are found to be zero, the frame is inertial, or else non-inertial.

S41 *I have another query. Each day we see the sun moving across the sky. So relative to the earth the sun goes around the earth, while relative to the sun the earth rotates about its axis. Why, then, do we insist that the rotation of the sun is apparent, and it is indeed the earth that rotates?*

T41 This is just an application of the ideas already discussed. Rotation, unlike uniform motion, is not relative; it is an absolute idea. That is, there is a way to deduce whether it is the sun or the earth that is actually rotating.

Take the sun's frame of reference and consider the motion of the earth and other planets. This will be describable by the II law of Newton (combined with the law of gravitation) with no (or negligible) addition of pseudo-force terms. Thus the sun's frame is inertial. Now take the earth's frame and consider the motion of the sun and other planets. One finds that one needs to include in the II law of motion pseudo-force terms which contain an angular velocity parameter. This parameter is the same whether we consider the motion of the sun or some planet; it must be interpreted as the angular velocity of the frame itself, i.e. that of the earth. Thus the earth's frame is non-inertial; it is the earth that actually rotates, not the sun.[1]

S42 *But we do consider the earth's frame as inertial for most terrestrial experiments. How can the same frame of reference be inertial for one purpose and non-inertial for another purpose ?*

T42 It is a question of the relative magnitudes of the pseudo-force terms and the material force terms which appear in the II law of motion for a non-inertial frame. You see, the ratio F_p/F depends not only on the parameters of the frame such as α, ω etc., but also on other things like mass, position and velocity of the body and characteristics of material forces.

For astronomical observations relative to the earth's frame, the pseudo-forces cannot be neglected. Indeed, they mainly account for the observed daily rotation of the stars. For most (but not all) terrestrial experiments, however, the ratio F_p/F is negligible. For these situations, therefore, the earth's frame can be considered an approximately inertial frame.

In short, inertial or non-inertial character of a frame is a matter of degree, and depends on the realm of your experiments. An ideal inertial frame, for which all the acceleration / rotation parameters are strictly zero, does not exist in nature. The frame of reference of distant fixed stars is a good approximation to an ideal inertial frame.

S43 *You have emphasized throughout the discussion that the inertial or non-inertial character of a frame can be ascertained only empirically. But surely, there must be some deeper*

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[1] We have ignored in this discussion the angular velocity of revolution of the earth about the sun.

reason why, in nature, some frames turn out to be inertial and others non-inertial. For example, in Newton's pail experiment, why is it that the laboratory frame is inertial and the frame of the rotating pail is non-inertial, and not the other way ?

T43 This is an important question, and I wish I could give you a definitive answer. The famous Mach's principle is a response to a question of this kind. In the pail experiment, if the inertial observer looks out to the sky, he will find the distant stars fixed. On the other hand, the non-inertial observer finds the distant stars rotating. Thus we find that a (local) inertial frame happens to be one in which the distant stars in the universe do not rotate.

Mach converted this coincidence into a principle. He conjectured that, in a proper cosmological theory, local inertial or otherwise character of a frame is determined by matter distribution in the entire universe. In other words, in this view the pseudo-forces in the frame of the rotating pail are actually material in origin -- they arise from distant matter in the universe. I suggest you consult a more advanced text to get a proper view of these rather subtle matters.

Examples of non-inertial frames

S44 *Let us turn to more mundane matters. Could you give me some examples where it is more convenient to consider non-inertial frames ?*

T44 Consider a box lying on the floor of a train that is moving with an acceleration **a**. Can you explain the motion of the box from the viewpoint of a ground observer ?

S45 *That is simple. The weight of the box is cancelled by the normal reaction on it by the floor. The horizontal acceleration of the box is caused by the force provided by the engine driving the train.*

T45 The last statement of yours is very vague physics-wise. We emphasized in chapter 4 that in mechanics we should invoke only two kinds of material forces : the gravitational force and contact forces. So, instead of making a vague layman-like statement that the engine gives a force to the box, you should say that the horizontal acceleration of the box is caused by the frictional force between the box and the floor.

S46 *What ? Frictional force in the same direction as the direction of motion ? I thought it always opposed motion.*

T46 Static friction opposes impending relative motion between surfaces in contact. In the absence of friction, for a ground observer, the box will stay where it is (I law), while the floor of the train will slide by. To oppose this impending relative motion, the frictional force drags the box with the same acceleration **a** as that of the floor. Relative to the floor, therefore, the box remains at rest.

It is much easier to see this relative to the frame of reference of the accelerated train. This frame is non-inertial and therefore the box is subject to a pseudo-force $-m\mathbf{a}$. To oppose the impending motion due to this force, an equal and opposite static frictional force comes into play keeping the box stationary.

S47 *Any other example of a non-inertial frame in practice ?*

T47 Consider the uniform circular motion of a stone tied to the end of a string, relative to the laboratory frame of reference which is given to be inertial. For circular motion, kinematically, $a = v^2/r$ and it is directed inward. This centripetal acceleration is caused by a material force namely tension **T** in the string. Therefore, according to the II law

$$T = \frac{mv^2}{r}$$

Notice, nowhere have I used the word centrifugal force above. People sometimes describe this situation as : 'The centrifugal force on the stone balances the centripetal force of tension keeping the stone in equilibrium'. This explanation of motion of the stone relative to the laboratory frame is wrong on two counts. First, in the laboratory frame, the stone is not in equilibrium. Second, a centrifugal force is a pseudo-force which should never be invoked in an inertial frame.

S48 *But the explanation is met so often: there must be something to it.*

T48 It is a correct explanation in a non-inertial frame which is rotating with the stone and has its origin at the centre of the circle. The angular velocity of this frame is constant - the same as that of the stone. Also, the Coriolis force on the stone is zero since it is

at rest in this frame. Thus, the only pseudo-force on the stone is the centrifugal force and hence the equation of motion for the stone in this frame is

$$m\,\mathbf{a} = \mathbf{F} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) .$$

Now, relative to the rotating frame the stone is in equilibrium, so that $\mathbf{a} = 0$. The material force continues to be the tension **T** whereas the magnitude of the centrifugal force is given by $m\omega^2 r$. Thus the above equation reduces to that in T47.

'Feeling' or 'experiencing' a force

S49 *You said that the centrifugal force is not a real material force. But one does experience it in practice, on a merry-go-round for example. How do you explain it ?*

T49 First of all, banish any idea that pseudo-forces are not 'real' forces. In a rotating frame, for example, centrifugal force is as real as any material force. The term pseudo-force is meant merely to emphasise the fact that it does not arise due to any material agency, but arises due to the rotation/acceleration of the frame itself.

On a merry-go-round, the centrifugal force on a man is balanced by the contact forces of friction and the reaction of the sides of the seat. These latter forces act only at the surfaces of the body in contact with the seat. Other parts of the body would fly off in the instantaneous tangential direction, like any other object let loose, if the forces constraining them were 'switched off'. The constraining forces acting on various parts of the body are generated in the muscles, bones etc., and the forces exerted by the muscles and bones etc. have their origin in the stresses developed in them. These body stresses are associated with our 'experience' of the centrifugal force. The net result is that each part of our body is subject to an inward material force which balances the outward centrifugal force.

The interesting thing is that the brain interprets the body stresses as our experience of an outward force. Thus we 'feel' the force in the direction of impending motion.

S50 *How does a bystander view the situation ?*

T50 In the ground's (inertial) frame of reference, there is no centrifugal force. The inward material force on each part of the body causes the observed circular motion in accordance with the II law.

S51 *Suppose a person is standing on a rotating disk: imagine that the friction is suddenly 'switched off'. There is then no inward material force and he will be pushed outward by the centrifugal force. How will a bystander, watching this motion, explain it without invoking a centrifugal force ?*

T51 Be careful. What is outward motion† in the rotating frame is a straight-line motion, along the instantaneous tangential direction, to a bystander on the ground. He easily explains this motion as due to inertia (the I law) and does not need to invoke any force.

S52 *Suppose I invert the problem. Relative to the person in the merry-go-round, the outsider will seem to move in a circle in the opposite sense. Now, there is no force on him because he is outside the merry-go-round. How can the person in the merry-go-round explain the circular motion of the outsider relative to him ?*

T52 You have made a serious error here. I warned you early in this chapter not to think that a frame of reference is bounded and localized. The frame of reference rotating with the merry-go-round does not terminate at its boundary. It is wrong to think that in this rotating frame objects on the merry-go-round only experience pseudo-forces. In this frame all objects in the universe experience pseudo-forces.

S53 *I realize the error. But I am still confused about one thing. In the rotating frame acceleration of the outsider is centripetal i.e. inward, while the centrifugal force on him is outward. There seems to be a mismatch of directions.*

T53 The error lies in ignoring another pseudo-force, the Coriolis force. Go back to T28 [Eq.(20)]. If an object is stationary in a rotating frame, centrifugal force is the only pseudo-force acting on it. But for an object that is in motion relative to a rotating frame, there is an additional velocity-dependent term : the Coriolis force. Since the outsider is in motion relative to the merry-go-round, he will be subject to both

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† It should be noted that the outward motion is governed not only by the centrifugal force but also by the velocity-dependent Coriolis force. Thus, in the rotating frame, motion with $F = 0$ is complicated and not purely radial as is sometimes erroneously claimed.

centrifugal and Coriolis forces. It turns out, in this case, that the latter is twice in magnitude and directed inward. The net force on the outsider, therefore, is inward as you required. (See Box 5.3.)

Box 5.3

Consider a merry-go-round rotating with a constant angular velocity ω. A person of mass m is standing on the ground. Let r denote the position vector of the person. Since the person is stationary on the ground, his velocity v relative to the ground's frame (S) is zero. Eq.(9) gives his velocity v' relative to the S' frame rotating with the merry-go-round $v' = -\omega \times r$. (See Fig.5.9.)

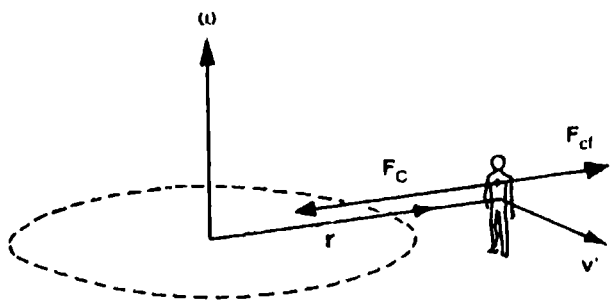

Fig.5.9 The platform rotates with an angular velocity ω. v' is the velocity of the bystander relative to the rotating frame of the platform.

The centrifugal force on the person is $F_{cf} = -m\omega \times (\omega \times r)$.

The Coriolis force on the person is $F_C = -2m\omega \times v' = 2m\omega \times (\omega \times r)$.

Clearly, the centrifugal force points away from the centre whereas the Coriolis force points towards the centre. Also, the magnitude of the Coriolis force is twice that of the centrifugal force. Thus the net force on the bystander has a magnitude $m\omega^2 r$, the same as the magnitude of the centrifugal force, but points inward.

S54 *You said that, relative to the rotating frame, pseudo-forces act on the man on the ground as well. Yet we know that the man does not experience any such force. How do you explain that ?*

T54 I explained in T49 that the stresses developed in our body, in order to oppose any impending relative motion between different parts of our body, are associated with our 'experience' of a force. And the direction of the force 'felt' is the direction of the impending motion. Now, if the impending motion is not opposed, no stresses are produced in the body and no force is felt ! An example will clarify the point.

An astronaut in outer space far away from all bodies feels weightless because there is no force acting on him; there is no impending motion between different parts of the body and hence no efforts on his part to oppose it.

While standing on the surface of the earth, the normal force by the ground acts on the feet of a person. Other parts of the body would fall downwards under gravity if there were no forces constraining them together. These forces have their origin in the stresses developed in the body. These stresses are associated with our 'experience' of weight. Note that the net force on each part of the body is zero; but the brain interprets the body stresses as a force in the downward direction -- the direction of impending motion.

Now consider an astronaut in a satellite orbiting the earth. The gravitational force on him is not zero; in fact that is the force which keeps him in the orbit, along with the satellite and everything else in it. Why, then, does he 'feel' weightless ? Because all parts of his body are freely falling under the force with the same acceleration; there is no relative motion between the different parts that he needs to oppose. Consequently, no stresses are produced in his body and he 'experiences' no force.

Returning to your question, the fact that a bystander does not 'feel' any force (apart from his weight) can be explained both by a ground observer and a rotating observer. For a ground observer, of course, there are no pseudo-forces on the man. Relative to the rotating frame, there <u>are</u> pseudo-forces acting on the bystander. But all parts of his body are rotating freely under their action with the same acceleration and there is no relative motion between the different parts that he needs to oppose. Hence he does not 'feel' the pseudo-forces.

S55 *One more query. Pseudo-forces or frame-dependent forces are also called 'inertial forces'. What is the origin of this term ?*

T55 There is a simple reason behind that term. As you know, the constant m appearing in Newton's II law $\mathbf{F} = m\mathbf{a}$ is called the inertial mass of the body since that is a

measure of the inertia of the body (the property of resisting change in its state of motion). This same constant m also appears in pseudo-forces in a non-inertial frame. The constants appearing in the material forces may, however, be different. In particular, the mass constant that appears in the gravitational force is the gravitational mass of the body. It is another matter that the gravitational mass turns out to be proportional (or equal by choice) to the inertial mass of a body. If the body is charged, material forces may include other constants like charge, magnetic moment etc. But whatever the constants appearing in material forces, the pseudo-forces always contain only the inertial mass constant of the body. It is to highlight this fundamental difference between material forces and pseudo-forces that the latter are called 'inertial forces'.

Beware of possible terminological confusion here. The term 'pseudo-forces', as we saw, can give you a wrong impression that they are fictitious, not real, even in a non-inertial frame. The term 'inertial forces' can sometimes erroneously suggest that they are forces in an inertial frame ! The term 'frame-dependent forces' is perhaps the least confusing of all.

Relativity of Einstein

S56 *A final question on these matters. We have talked of relativity in Newtonian mechanics. In what ways does it differ from the relativity of Einstein?*

T56 I can only answer this qualitatively. A detailed discussion of Einstein's relativity is beyond the scope of this book. As we have seen, Newtonian theory satisfies the following principle of relativity:

The laws of mechanics are the same for different inertial observers in uniform relative motion.[1]

Note the underlined word : mechanics. The basic theory of electromagnetism had not yet arrived. When it arrived two centuries later, a perplexing thing happened.

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1 This implies that the laws do not require a frame of absolute rest. Newton was aware of this; yet, paradoxically, he still believed in a frame of absolute rest -- the ether. The ether was thought of as something that mediated the gravitational force. Other reasons may be partly aesthetic and partly a product of the time : it was difficult in those times to imagine a cosmic order based on equity !

Maxwell's equations of electromagnetism contained a constant that was identified with the velocity of light in vacuum. Now in Newtonian mechanics, a velocity cannot have the same value in two frames in relative motion. Thus the laws of electromagnetism, involving one definite value of velocity, seemed to pertain to one particular frame. That is, the laws did not satisfy Newtonian relativity, and seemed to single out a privileged frame among the class of inertial frames, which could be taken to be the frame of absolute rest : ether.

In short, it seemed that the laws of mechanics did not require ether while the laws of electromagnetism did. Einstein was not willing to accept this unsymmetric state of affairs.[1] He rejected the idea of ether and postulated that the laws of physics, whether of mechanics or of electromagnetism are identical for all inertial observers. This is Einstein's first postulate : the principle of (special) relativity:

(I) The laws of physics are the same for all inertial observers in uniform relative motion.

Now if you believe Maxwell's equations to be the correct laws of electrodynamics, postulate (I) implies that the constant c in these equations i.e. the velocity of light should be the same relative to all inertial frames. This conclusion is totally against our common sense. It violates the usual velocity addition rule that we use so often :

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{V} .$$

S57 *Then how did Einstein get around the problem ? How did he explain the constancy of the velocity of light relative to different frames ?*

T57 He did not. His genius lay in turning the problem around. He took the constancy of c as another postulate of his theory :

(II) The velocity of light in vacuum is the same for all inertial frames i.e. it is independent of the relative motion between the source and the observer.

and used it to modify the velocity addition rule so that the latter is consistent with (II)! Now the velocity addition rule is based on our very elementary and intuitive ideas of

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1 Also, careful experiments for detection of ether had yielded negative results. However, it is not very clear whether Einstein was aware of these, or to what extent they motivated his ideas.

of distance, time etc. -- ideas so primitive that nobody ever thought of questioning them. So Einstein was led to alter our conceptions of space and time.

S58 *I have heard that last statement so often. What is it supposed to mean?*

T58 Let us first spell out our intuitive conceptions of space and time. First, we feel that the time of an event has little to do with the external observer. Of course, different observers may differ in the zero settings of their clocks. So, strictly, we should say that the time difference between two events is independent of the frame of reference. In particular, if two events are simultaneous to one observer, they will be so to any other observer. Likewise, the dimensions of an object, e.g. the length of a rigid rod seems to be its intrinsic property that has nothing to do with its motion i.e. it is independent of the frame of reference. These intuitive ideas underlie the Galilean transformations [Eq.(6)] that we obtained early in this chapter.

S59 *So what is wrong with these intuitive ideas?*

T59 We have seen that the velocity addition rule, in particular, is a corollary of Galilean transformations. Since this, as we saw, is inconsistent with Einstein's theory, he had no choice but to reject Galilean transformations, and replace them by new transformations, based on the constancy of velocity of light relative to all inertial frames. These new transformations are known as Lorentz transformations. I quote them without proof in Box 5.4. You will need to consult a standard text on relativity to understand them properly.

Box 5.4

Lorentz Transformations :

Let S be an inertial frame and let S' be another frame moving uniformly relative to S with a velocity v along say the x-axis of S. Let the axes of the S' frame be parallel to those of S as shown in Fig.5.10. Let O and O' coincide at t = t' = 0.

Suppose that an event occurs at (x, y, z) at time t relative to the S frame. Relative to the S' frame, the same event occurs at time t' at (x', y', z'). The space-time coordinates of the same event relative to the two frames are related to one another by the equations :

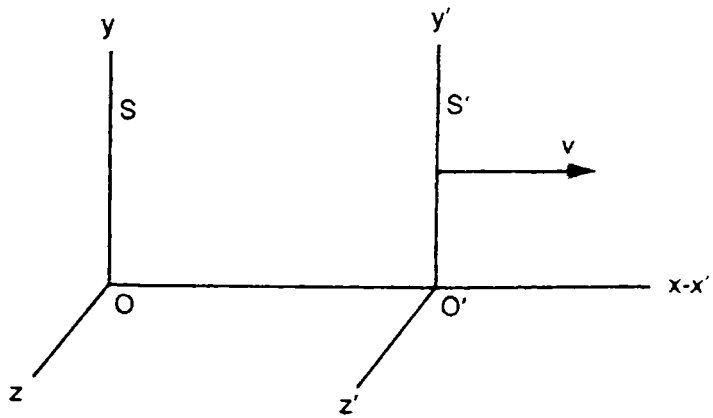

Fig.5.10 The frame S' moves with a uniform velocity v relative to S along the common x-x'-axis.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} ,$$

$$y' = y ,$$

$$z' = z ,$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} .$$

These relations are known as Lorentz transformations.

Four important kinematic implications of Lorentz transformations are worth keeping in mind .

1) Relativity of simultaneity :

If two events (at different locations) are simultaneous in a frame S, they are not so in a frame S' in relative motion. Simultaneity, therefore, is no longer absolute as in Newtonian relativity, but is a relative concept.

2) Length contraction :

Length determination, as we know, requires simultaneous location measurements. But if two observers do not agree on simultaneity, they will naturally not agree on their measured lengths. If ℓ_0 is the length of a rigid rod measured when it is at rest relative to an observer, and ℓ its length while it is moving along the direction of its length with a velocity v (relative to the same observer), then

$$\ell = \ell_0 \sqrt{1 - v^2/c^2}$$

In other words, the length of a moving rigid rod is shorter than its length at rest.

3) Time dilation :

This refers to the time interval Δt between two events at the **same** location in one frame (S). In another frame (S') which is moving uniformly relative to the first, the time interval between the same events, $\Delta t'$, is measured to be greater :

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

4) Velocity addition rule :

The ordinary velocity addition rule is no longer true; the relativistic velocity addition rule ensures that c is the same for all frames, as it should be for consistency, since that is the starting axiom employed in the derivation of Lorentz transformations.

If u is the velocity of an object relative to S, and u' the velocity of the same object relative to S', then

$$u' = \frac{u - v}{1 - uv/c^2} .$$

You can check that for u = c, u' = c.

S60 *Do Newtonian laws of motion remain unchanged in special relativity ?*

T60 You must get this clear.

Newtonian laws of motion are invariant, i.e. they retain the same form under Galilean transformations, but not under Lorentz transformations.

The laws of electromagnetism (Maxwell's equations) are invariant under Lorentz transformations, but not under Galilean transformations.

So, if the laws of physics are to be consistent with special relativity based on Lorentz transformations, Maxwell's equations do not require any modification but Newton's laws of motion do.

S61 *What are these modified laws of motion in special relativity ?*

T61 Newton's first law of motion is the same as before; it just defines an inertial frame. The II law of motion in the form

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \tag{21}$$

is still true, but the relativistic definition of momentum must be carefully noted :

$$\boldsymbol{p} = \frac{m_0 \boldsymbol{v}}{\sqrt{1 - v^2/c^2}} . \tag{22}$$

One way of interpreting this new definition is to lump m_0 and $\frac{1}{\sqrt{1 - v^2/c^2}}$ together and call the product as 'moving mass' of the particle

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} . \tag{23}$$

If v in this equation is zero, $m = m_0$. m_0 is, therefore, called the rest mass of the particle. Eq.(23) is the basis of the often-quoted phrase : mass of a body increases with speed. The important thing to note is that Eqs.(21) and (22) give

$$\boldsymbol{F} = \frac{d}{dt}\left(\frac{m_0 \boldsymbol{v}}{\sqrt{1 - v^2/c^2}} \right) .$$

i.e. the II law in the form $\mathbf{F} = m\,\mathbf{a}$ is <u>not</u> correct in special relativity. In particular, note that the force is not necessarily in the same direction as $d\mathbf{v}/dt$.

To sum up, the II law in the form $\mathbf{F} = d\mathbf{p}/dt$ is true in both Newtonian and relativistic mechanics. In the former $\mathbf{p} = m_0\,\mathbf{v}$, while in the later $\mathbf{p} = m\,\mathbf{v}$ where m is given by Eq.(23).

S62 *What about the III law in relativity ?*

T62 The situation is a bit more involved in regard to the III law. In the Newtonian picture, mutual forces between bodies (e.g. gravitational force, coulomb force etc.) located any distance apart are exerted instantaneously. This is against special relativity (and fact), since no signal can travel faster than the speed of light. Thus in a proper relativistic domain, e.g. for the mutual force between electrons moving at high speeds, the III law is not valid. Of course, for ordinary distances and speeds the times taken by signals to communicate forces are negligible and the III law is approximately valid.

For contact forces, obviously, there is no problem of the above sort, and we can continue to use the III law in relativistic mechanics.

6 Energy and Momentum

Work-Energy theorem

S1 *In physics books we often see the statement : "energy is the capacity to do work." What is this supposed to mean ?*

T1 To understand the phrase, it is not much use pondering over its literal meaning; rather let us try to see the physical idea behind it.

You know that the final velocity of a body, under the action of a force, depends on the initial velocity of the body, the magnitude and direction of the force and the time interval or the distance over which the force acts on the body. If the quantity of interest is the magnitude of the final velocity, we can do the following :

If the force is constant, we combine the II law $\mathbf{F} = m\,\mathbf{a}$ with the kinematical relation $v_f^2 - v_i^2 = 2\,\mathbf{a}\cdot\mathbf{s}$, valid for constant acceleration, to obtain

$$\boldsymbol{F}\cdot\boldsymbol{s} = m\,\boldsymbol{a}\cdot\boldsymbol{s} = \frac{1}{2}\,m\,(v_f^2 - v_i^2)$$

More generally, if $\mathbf{F}$ is varying, we have

$$\boldsymbol{F}.d\boldsymbol{s} = m\,\frac{d\boldsymbol{v}}{dt}.\boldsymbol{v}\,dt$$

$$= \frac{m}{2}\,d\,(\boldsymbol{v}.\boldsymbol{v})$$

$$= d\,(\frac{1}{2}\,m\,v^2)$$

which may be integrated as follows :

$$\int_i^f \mathbf{F}.d\mathbf{s} = \int_i^f d\left(\frac{1}{2} m v^2\right)$$

$$= \frac{1}{2} m (v_f^2 - v_i^2)$$

The notions of work and kinetic energy are introduced at this point. The right hand side of the equation is a change in the quantity : half the mass times the square of the speed of the body. We *call* this the kinetic energy of the body and denote it by T. The left hand side is *called* the work done by the force **F** on the body over its path from i to f, and is denoted by W. We then write the above equation as

$$W = T_f - T_i .$$

This is known as the work-energy theorem. It says that the work done by the net external force on a body equals the change in its kinetic energy. As we have seen, the theorem is a direct consequence of the II law of motion.

S2 *Does that mean I can always use the work-energy theorem, in place of the II law, while solving problems in mechanics ?*

T2 The theorem is based on the II law; but notice that the II law is a vector equation (which is equivalent to having three scalar equations), while the work-energy theorem is a single scalar equation. Thus, the latter does not have the full dynamical information contained in the II law. So, when you can solve the equation **F** = m **a** for a problem, you have all the possible information about the motion. Sometimes, however, solving the equation of motion may be difficult or unnecessary. Important information about the motion, in such situations, can be obtained from energy considerations alone. (See S30-T30.)

S3 *I am clear that work gets converted into kinetic energy. But in what sense is kinetic energy the 'capacity' to do work?*

T3 Suppose a block of mass m is moving with velocity v, so its kinetic energy is $\frac{1}{2}mv^2$. Suppose, further, that it hits an elastic spring clamped on a wall. As the spring gets compressed, it exerts a force F_{bs} on the block. The block exerts an equal and opposite force $F_{sb} = - F_{bs}$ on the spring. The velocity of the block decreases progressively

because of the retarding force $\mathbf{F}_{bs}$, finally (momentarily) becoming zero. The work W_{sb} done by the block, during the course of retardation, on the spring is given by

$$W_{sb} = \int_i^f F_{sb}\, dx$$

$$= -\int_i^f F_{bs}\, dx = -(T_f - T_i).$$

where T_i and T_f refer to the initial and final kinetic energy of the block respectively. Since $T_i = \frac{1}{2} mv^2$ and $T_f = 0$, the work done by the block before coming to rest is $W_{sb} = \frac{1}{2} mv^2$

Thus, the kinetic energy is indeed the amount of work that an object can do before coming to rest.

When is the work of a force zero ?

S4 *It is always emphasized that a force does work only when the point of application of the force moves. Yet when I press a wall with some force, or hold an object in my hand, I do seem to do work; else why should I feel tired after some time ?*

T4 You must clearly distinguish between the technical meaning of the term 'work' in physics and the meaning of this word in ordinary language. Physics-wise, the force on the wall does no work, since the wall does not move. Likewise, the force due to gravity, or the reaction on the object by the hand, does not do any work if the object is stationary. (In practice, it keeps shaking slightly.)

What then explains the feeling of tiredness ? You see, our body is not rigid: it is a complex system which is deformable within certain limits. When you press against a wall, or provide reaction to an object in your hand, the body configuration is altered slightly from what it is in the relaxed state. To bring and sustain it in this altered state, certain complex processes (chemical reactions) take place in your body, resulting in depletion of nutrients and production of heat. This causes tiredness.

Anthropomorphic† considerations are a very common source of confusion in understanding the basic concepts of mechanics. To avoid this pitfall, in particular problems, it is best to replace humans by mechanical devices ! The device can be arranged to provide the same force on the external object as the human does. The physics regarding the external object remains the same. And the internal details of the force agency, which are irrelevant for the problem, do not mislead you.

Thus, the force on the wall provided by you can be modelled by say a suitable arrangement of rigid rods and springs. It is then seen that the system of rods and springs provides a force but, as long as the configuration is static, it does no work. In providing the force, stresses are developed in the mechanical assembly. However, the state of the assembly, in the stressed condition, is an equilibrium state and, unlike in the case of human body, no energy need be expended in order to keep it in that state.

S5 *Is it possible that there is a net force on a body in motion, and yet no work is done by the force ?*

T5 Yes, of course. Whenever a force is normal to the displacement of the body, the force does no work. In uniform circular motion there is a net force on the body in the radially inward direction whereas the displacement at every point is in the tangential direction. Hence no work is done by the force and the kinetic energy of the body is constant.

S6 *Is circular motion the only example of this type ?*

T6 No. All that is required is that at every instant the force **F** be normal to the velocity at that instant. An important example is the motion of a charged particle in a magnetic field. The magnetic force on a charge is given by

$$\boldsymbol{F} = q\, \boldsymbol{v} \times \boldsymbol{B}$$

Clearly, the magnetic force is always perpendicular to the velocity and hence it does no work.

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† human-like, pertaining to humans.

The fact that the force is always perpendicular to velocity, by no means implies that the trajectory of the particle is necessarily circular. It is circular if the magnetic field is uniform and the initial velocity of the charged particle is normal to the field.

In case the initial velocity has a component parallel to the direction of the uniform magnetic field, that component remains unaffected by the field. The net effect is that the particle has a uniform translational motion in the direction of the field, superposed on a uniform circular motion in a plane normal to the field. The motion of the particle, then, is not circular but helical. For non-uniform magnetic field the trajectory can be even more complicated. The work done by the magnetic force, nevertheless, is zero in every case.

In the above examples, the work over <u>any</u> segment of the path of the particle is zero. We shall see later that for the so-called conservative forces the work over an arbitrary path is not zero, but that over a closed path is zero.

S7 *Is it possible that the net external force on a body is zero, and yet some work is done on it ?*

T7 For a single point particle this is obviously not possible, but for extended bodies, or in general for a system of particles, this <u>is</u> possible.

For a rigid body, for example, the net external force may be zero and yet the net torque may not be zero. The torque does work over the angular displacement of the body, thereby changing the rotational kinetic energy of the body.

For a deformable body, the points of application of different forces on the body may move through different displacements. Thus it is possible that, even if $\sum \boldsymbol{F}_i = 0$, we have $\sum \boldsymbol{F}_i \cdot \Delta \boldsymbol{s}_i \neq 0$. A familiar example is a spring compressed by two equal and opposite forces at its ends. Clearly, the net force on the spring is zero, but the net work done by the forces is not zero.

Note that if the net force is zero, the acceleration of the centre of mass of the system is zero and thus there is no change in the translational kinetic energy of the centre of mass of the system.

S8 *I have another query. Is the work done by A on B equal in magnitude but opposite in sign to the work done by B on A ?*

T8 The III law of Newton says that the force on A by B is equal and opposite to the force on B by A. There is no similar statement possible for the quantity : work. Take care not to be misled by this pseudo-III law for work. Let us get this clear by an example.

Consider the fall of a stone under the gravity of the earth. The earth exerts a force mg on the stone; the stone exerts an equal and opposite force on the earth (III law). However, the earth is much more massive than the stone. So, for the same magnitude of force, its displacement is negligible compared to that of the stone. Thus the work done by the earth on the stone is mgh, where h is the displacement of the stone; but the work done by the stone on the earth is negligible.†

S9 *I understand your point. But one does seem to use this kind of rule at many places. For example, in thermodynamics one often says that work done by the system on the surroundings is negative of the work done on the system by the surroundings.*

T9 That statement is correct in the context in which it is made. Consider a gas in a cylinder with a movable piston. Suppose the piston is pushed down with some acceleration. In this case, work done by the outside agency on the piston is positive, while the work done by the system (gas) on the piston is negative. The two quantities are not equal in magnitude; their sum, in fact, equals the increase in kinetic energy of the piston, according to the work-energy theorem.

In equilibrium thermodynamics, however, the piston is imagined to move quasi-statically. That is, at every stage the piston's kinetic energy is zero. In this case the two works referred to above are negative of each other.

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† The (negligible) displacement of the earth, in fact, is in the same direction as the force on it due to the stone. The work done by the stone on the earth is, thus, positive. The work done by the earth on the stone is, likewise, positive. The quantities, clearly, are neither equal in magnitude nor opposite in sign.

Universality of work-energy theorem

S10 *Is the work-energy theorem true in non-inertial frames ?*

T10 Yes, provided we take into account the work done by the pseudo-forces also. Take, for simplicity, a non-inertial frame that has only translational acceleration $\boldsymbol{\alpha}$. In this frame the II law takes the form

$$m \frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} - m\boldsymbol{\alpha} .$$

Following the steps in T1, we obtain work-energy theorem in a non-inertial frame as

$$\frac{1}{2} m (v_f^2 - v_i^2) = \int_i^f \boldsymbol{F}\cdot d\boldsymbol{r} - m \int_i^f \boldsymbol{\alpha}\cdot d\boldsymbol{r} .$$

Thus, in a non-inertial frame, the kinetic energy of an object changes not only due to the work done by the material forces but also by the work of the pseudo-forces. In particular, even in the absence of material forces, the kinetic energy of an object changes due to the latter.

S11 *Is the work-energy theorem true in special relativity ?*

T11 Yes, in a suitably generalized form. As we saw in chapter 5, the II law in the relativistic domain is

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) , \qquad (1)$$

where

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} . \qquad (2)$$

Multiplying by $d\boldsymbol{r} = \boldsymbol{v}dt$ on the two sides of Eq.(1) we write

$$\boldsymbol{F}\cdot d\boldsymbol{r} = \frac{d}{dt}(m\boldsymbol{v})\cdot\boldsymbol{v}\, dt = \left(\frac{dm}{dt}\boldsymbol{v}\cdot\boldsymbol{v} + m\frac{d\boldsymbol{v}}{dt}\cdot\boldsymbol{v} \right) dt . \qquad (3)$$

Now, Eq.(2) may be written in the form

$$m^2c^2 = m^2v^2 + m_0^2c^2 .$$

Differentiation of this equation with respect to time gives

$$\frac{dm}{dt} c^2 = \frac{dm}{dt} v^2 + m\, \boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} .$$

Substituting this in Eq.(3), and integrating from the initial state i to the final state f, we get

$$\int_i^f \mathbf{F} \cdot d\mathbf{r} = m_f c^2 - m_i c^2. \qquad (4)$$

Since the left hand side is the work done, it may be natural to identify the right hand side as the change in kinetic energy $T_f - T_i$. It will be erroneous, however, to take kinetic energy $T = mc^2$. We expect kinetic energy of a body to be zero when its velocity is zero whereas $mc^2 \rightarrow m_0c^2$ as $v \rightarrow 0$. Thus, in special relativity, kinetic energy is defined as

$$T = mc^2 - m_0c^2 .$$

This suggests that we introduce a new physical quantity called total energy E defined as

$$E = mc^2 = T + m_0c^2 .$$

The total energy E includes the kinetic energy T and the so-called rest energy m_0c^2.

The work-energy theorem in relativity [Eq.(4)] is expressed in terms of E as

$$W = E_f - E_i .$$

S12 *What is the significance of this ? Is $E_f - E_i$ not the same thing as $T_f - T_i$?*

T12 Ordinarily the rest mass m_0 is a constant and $E_f - E_i$ is the same as $T_f - T_i$. However there are situations where this is not true. And that is when we come to a most dramatic consequence of special relativity.

Suppose two bodies, each of rest mass m_0, collide head-on with each other with equal and opposite velocities relative to a particular frame of reference S. Clearly, relative to S, the total momentum before collision is zero :

$$p_{before} = 0.$$

Also, the kinetic energies, and hence the total energies, of the two bodies are equal. If $T + m_0c^2$ is the energy of one body, the total energy of the system before collision is

$$E_{before} = 2\,(T + m_0c^2)\,.$$

Suppose, now, that the collision is perfectly inelastic so that the two bodies combine to form a single composite. Momentum conservation requires that the net momentum after collision be zero. Hence the velocity, and the kinetic energy, of the composite body must be zero. Total energy conservation then requires that

$$E_{after} = 2\,(T + m_0c^2)\,.$$

Clearly, the mass M_0 of the composite body must be such that

$$M_0c^2 = 2\,(T + m_0c^2)\,.$$

Evidently, $M_0 > 2m_0$.

We thus see that kinetic energy of the colliding bodies appears as increased rest mass energy of the composite body. This is the well-known interconvertibility of mass and kinetic energy in relativistic dynamics.

Path independence of work and potential energy

S13 *Let us turn to the notion of potential energy. It is supposed to be the energy associated with the position of a system or its configuration. What does that mean ?*

T13 Before I come to it, let me emphasize one thing. The definition of kinetic energy makes no reference to forces acting on an object. Likewise, the notion of work done by a force

is general: it makes no reference to the nature of the force. Consequently, the work-energy theorem is true, no matter what kind of forces act on the body.

The notion of potential energy, on the other hand, is applicable only to a class of forces called conservative forces. (The reason for naming them 'conservative' will be clear to you when we go up to T18) Thus, for example, one can associate potential energy with gravitational force, Coulomb force, spring force etc., but not with friction or viscous forces etc.

S14 *I am surprised. What is this special class of forces ?*

T14 Let me explain it with an example. Take the gravitational force that you are familiar with. This force has one very fundamental property that you must know. It is the following :

Consider the displacement of a particle from point 1 to point 2 in space under the influence of the gravitational force. It can be shown that the work done by the gravitational force on the particle depends only on the initial and final positions of the particle: it is independent of the path taken by the particle between the points 1 and

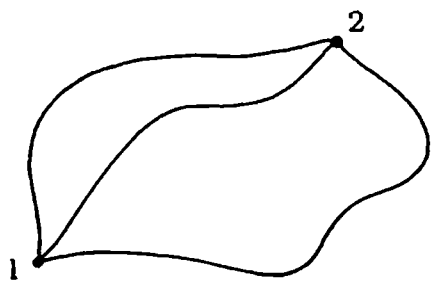

Fig.6.1 Different paths between points 1 and 2.

2. See Fig.6.1. An immediate corollary of this result is that the work done by gravitational force over a closed path is zero. You may consult any standard book for a proof of this result.

Any force for which a similar result holds true, i.e. any force for which work is independent of the path, is called a conservative force. Gravitational force, Coulomb force, elastic spring force are examples of conservative forces. Friction is a non-

conservative force. The notion of potential energy can be introduced for conservative forces only.

S15 *I am a little confused about this 'path-independence'. A particle under the action of a given force follows a definite path depending upon the initial conditions. What does one mean by taking different paths of the particle between two points ? An arbitrary path may not be a possible path of the particle for a given force. Isn't that so ?*

T15 What you must realize is that, although we are interested in the work done on a particle by a particular force, that may not be the only force acting on the particle. For example, a stone can be dropped from the top of a building to a point on the ground, or a person could carry it down via a winding staircase. The first path of the particle is under the gravitational force only; the second path is possible because the particle is subject to both the gravitational force and the appropriate force provided by the person. The point is that the work done by the gravitational force is the same for the two paths, or for any other path that you take between the same two locations.

More formally, if you prefer, the path independence may simply be taken as a mathematical property of the integral $\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}$ that follows from the particular form of the force.

S16 *I grant you this 'path-independence of work' property of gravitational force. What is so significant about it ?*

T16 It is precisely this property that leads to the notion of potential energy. It can be proved that work done on a body of mass m by the gravitational force due to a mass M at the origin is given by

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = -G\frac{Mm}{r_1} + G\frac{Mm}{r_2}.$$

Note that the right hand side depends only on the initial and final locations of mass m and not on the intervening path. If we define a quantity V that depends only on position by

$$V(\mathbf{r}) = -G\frac{Mm}{r}$$

then

$$\int_1^2 \mathbf{F}\cdot d\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) .$$

Thus the work done by the gravitational force on a particle during its displacement from $\mathbf{r}_1$ to $\mathbf{r}_2$ equals the difference between the values of the quantity V at the points $\mathbf{r}_1$ and $\mathbf{r}_2$. We can give this quantity some name : it is called gravitational potential energy.

S17 *Why is it called 'potential energy' ? In what sense is it energy ? The way you have arrived at it, it is just some position-dependent quantity.*

T17 I have stressed often that in physics one should not worry much over what a particular thing is called; rather one should concentrate on the precise definitions of quantities and any facts of physical significance. Many times the names of quantities could be misleading. In this particular case, however, the name 'potential energy' makes good sense.

First note the obvious. $V(\mathbf{r})$ has the correct dimensions, the same as those of work, to qualify as energy. Next, if work done by the gravitational force, in displacing the particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ is positive, V decreases [$V(\mathbf{r}_2) < V(\mathbf{r}_1)$]. If, on the other hand, an external agency displaces the particle against the gravitational force, so that work done by the gravitational force is negative, the quantity V increases [$V(\mathbf{r}_2) > V(\mathbf{r}_1)$]. In other words, work done by the external agency against the gravitational force goes to increase the quantity V. Hence V does seem to be some kind of 'stored-up energy'.

We shall see later that during motion under a gravitational force, the loss of V appears as gain in the kinetic energy T of the particle. The qualifier 'potential' is quite clear. $V(\mathbf{r})$ is a kind of (location-dependent) stored-up energy of the particle which has a potential of converting itself into kinetic energy (or work) when the particle is let go under the gravitational force.

S18 *I have a small query here. Why did you put a negative sign in your definition of V(r) ?*

What is wrong in defining V(r) = + (GMm)/r ?

T18 You will appreciate it when I prove an important result

For any kind of force, according to the work energy theorem,

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = T_2 - T_1$$

Remember, in this integral, F is the net force on the body.

For gravitational force, we have seen that

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = V_1 - V_2 .$$

If gravitational force is the only force acting, the two equations above give

$T_1 + V_1 = T_2 + V_2$. *i.e.* $T + V = constant$.

Now, about your query regarding the sign. Of course, one could define V(r) as + (GMm)/r. Then, obviously, we would get T – V = constant. Nothing wrong. We could have called T – V as net energy or something. But you will agree, it is much more cute to define V, as we have done, so that we get T + V = constant. That enables us to introduce a quantity E = T + V, and call it the total energy of the particle. The result above shows that the total energy thus defined remains constant in the actual motion of a particle under gravitational force

Now we see why gravitational force is called a 'conservative' force. For this force, there is a quantity (called potential energy of the particle) whose sum with kinetic energy is conserved during motion under the force.

S19 *Does the discussion above go through for any conservative force ?*

T19 Yes, it does. As already said, a force is conservative if $\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of the

path between the points $\mathbf{r}_1$ and $\mathbf{r}_2$, i.e. its value depends only on the positions $\mathbf{r}_1$ and $\mathbf{r}_2$. This shows that, for a conservative force, there exists a scalar function $V(\mathbf{r})$ such that

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) \,. \tag{5}$$

As in T18, this can be combined with the work-energy theorem to obtain

$$E = T + V = \textit{constant} \tag{6}$$

for motion under any conservative force. The earlier discussion on gravitational force is a particular example of this general argument with $V(\mathbf{r}) = -GmM/r$. Other conservative forces have different forms for $V(\mathbf{r})$.

S20 *I don't see how you have jumped to the particular form on the right side above in Eq.(5). Why not say $V(\mathbf{r}_1) + V(\mathbf{r}_2)$ or $[V(\mathbf{r}_1) - V(\mathbf{r}_2)]^3$?*

T20 Obviously, if the path is traversed in the reverse direction, the integral should change sign. So $V(\mathbf{r}_1) + V(\mathbf{r}_2)$ will not do. Further, path-independence also implies

$$\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} + \int_2^3 \boldsymbol{F}\cdot d\boldsymbol{r} + \;= \int_1^3 \boldsymbol{F}\cdot d\boldsymbol{r} \,.$$

Your cubed form will not satisfy this requirement. The only form possible is $V(\mathbf{r}_1) - V(\mathbf{r}_2)$. [The form $V(\mathbf{r}_2) - V(\mathbf{r}_1)$ is not really different -- it merely amounts to a redefinition of V as - V]. So there is no loss of generality in Eq.(5) above.

A direct proof of Eq.(5) can be given using the Stokes' theorem of vector calculus; see any standard text.

S21 *In practice, how does one check if a force is conservative or not ? Must we calculate the work done by the force over an arbitrary path between two points and check its path independence ?*

T21 We have defined a conservative force in terms of its integral property (namely path-independence of work). This is a global criterion as we say in physics. An equivalent local criterion (i.e. a condition at every point) is possible and is easy to work with.

Consider an infinitesimal displacement $\mathbf{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$. The work done by a force $\mathbf{F}$ over this displacement is

$$\mathbf{F}\cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz \ .$$

If the force is conservative, Eq.(5) gives

$$\mathbf{F}\cdot d\mathbf{r} = -\, dV$$

$$= -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \right) .$$

which gives

$$\mathbf{F} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \right)$$

i.e. $\mathbf{F} = -\nabla V$. Indeed you can take this to be a definition of a conservative force : a force is conservative if it can be expressed as the gradient of a scalar function. The 'path-independence of work' follows from this definition. Note, the choice of negative sign in $\mathbf{F} = -\nabla V$ has the same reason as explained in T18. In one dimension, $F = -dV/dx$.

S22 *It is said that the absolute value of potential energy has no physical meaning. How do you see that ?*

T22 That is obvious from the way we introduced the quantity $V(\mathbf{r})$ in Eq.(5). The left side of this equation, denoting the work, is a definite physical quantity; the force $\mathbf{F}$ and the points $\mathbf{r}_1$ and $\mathbf{r}_2$ determine its value completely. On the right side we have the difference between the value of $V(\mathbf{r})$ at the two points. I am, therefore, free to add any constant to the potential energy function $V(\mathbf{r})$; $V(\mathbf{r}_1) - V(\mathbf{r}_2)$ will still be the same. This means, $V(\mathbf{r})$ is undetermined to within an arbitrary additive constant. That is, the absolute value of potential energy at any one location is a matter of choice, and hence cannot have any physical significance. But once I choose some value for potential energy at any one point, the value of potential energy at all other points gets fixed; the difference in potential energy at any two points is not a matter of choice but is determined by the given force.

All this is clear from the relation $\mathbf{F} = -\nabla V$ also. ∇V and $\nabla (V + \text{constant})$ are the same. Clearly, given a force $\mathbf{F}$, V can be determined only to within a constant.

Lastly, note that there is no such arbitrariness in the definition of kinetic energy. Because of V, however, the total energy $E = T + V$ is also arbitrary to within an additive constant.

S23 *The discussion on conservative forces is clear to me. Can you consolidate it by additional examples ?*

T23 Consider first the simple one-dimensional example of the spring force $F = -kx$, where x is the extension of the spring. One way of seeing the conservative nature of the force is to obtain an appropriate potential energy function. It is easily seen that $V = \frac{1}{2} kx^2$ is the required function. (Use of this V in equation $F = -dV/dx$ leads to $F = -kx$.) We may add an arbitrary constant C to the potential energy: $V = \frac{1}{2} kx^2 + C$ also leads to the same force F. The arbitrariness in the value of potential energy may be removed by choosing a particular value for C. A convenient choice is $C = 0$, which makes $V = 0$ at $x = 0$. This choice of C, thus, corresponds to choosing zero of the potential energy at $x = 0$.

The electrostatic force between two charges, as given by the Coulomb's law, is another example of a conservative force. It can be verified that the potential energy function

$$V(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}} + C ,$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, leads to the vector form of the Coulomb's law.† Again, the potential energy is arbitrary to within an additive constant C and the arbitrariness is removed by choosing the zero of the potential energy at a convenient point. Clearly $r_{12} = 0$ is not a proper choice since $V \to \infty$ as $r_{12} \to 0$. If we choose $C = 0$, $V \to 0$ as $r_{12} \to \infty$. That is, we choose the electrostatic potential energy to be zero when the two point charges are infinitely separated.

In fact the choice of zero potential energy at infinite separation is customary and sensible for all the forces which tend to zero as the separation between the interacting bodies tends to infinity.

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† The force on charge 1, for example, has to be calculated as

$$\mathbf{F}_1 = -\left(\frac{\partial V}{\partial x_1}\hat{i} + \frac{\partial V}{\partial y_1}\hat{j} + \frac{\partial V}{\partial z_1}\hat{k} \right) .$$

S24 *Is it correct to say that if the force between two bodies is attractive, the sign of the potential energy is negative ?*

T24 For Coulomb and gravitational force (which change monotonically) with the choice of zero of potential energy at infinite separation, the statement you have made is correct. But it is obvious that your criterion cannot be true in general. The sign or the magnitude of the potential energy at any one point does not have any significance; it can be assigned any desired value, positive or negative, by appropriate choice of the arbitrary additive constant. For example, the harmonic oscillator potential energy $\frac{1}{2}kx^2$ is positive, though the force is always attractive.

The general criterion is : force between two bodies (at a given separation) is attractive if the change ΔV in the potential energy, with an increase in the separation distance between the bodies, is positive.

S25 *How should one decide whether a system will be bound or not for a given energy ?*

T25 The total energy of a system (E) is the sum of the potential energy (V) and the kinetic energy (T) of its constituents.

$E = T + V$.

V, and hence E, can be positive, negative or zero; the kinetic energy of a system, clearly, must always be non-negative.

Now, the total energy is a constant of motion for an isolated system. The specified value of E, and the form of V, determine the regions in which the kinetic energy of the system can be non-negative. Only these regions are accessible to the system.

Let us denote by $V(\infty)$ the potential energy of a system when the separation between the constituents is infinite. (See Fig.6.2.) For $E > V(\infty)$, evidently, $T > 0$ for infinite separation and the constituents of the system are free to move infinitely apart from each other. Such a system is said to be free.

On the other hand, if $E < V(\infty)$, $T < 0$ for infinite separation which is not allowed. Clearly, in this case, the constituents must always stay within a finite distance from each other. Such a system, whose constituents are confined to a finite region of

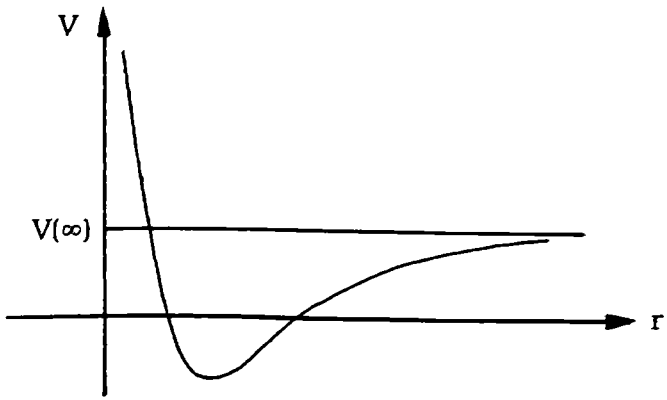

Fig.6.2 A potential energy graph.

space, is called a bound system.

S26 *You said earlier that friction is not a conservative force. How do you see that ?*

T26 It is easy to see that using the criterion of path-independence: in particular, I can check if the work done by friction over a closed path is zero.

Suppose an object is dragged around on a rough surface and brought back to the starting point. No matter which path you follow, the force of kinetic friction opposes the relative motion between the two surfaces at every point and, thus, always acts in a direction opposite to that of the displacement. Consequently

$$\oint \boldsymbol{F}_{friction} \cdot d\boldsymbol{r} < 0 \ .$$

Suppose, on the other hand, that an object is kept on a trolley and the trolley is moved around. The force responsible for the motion of the object is the force of static friction between the surfaces of the object and the trolley in contact, and this force of friction acts in the same direction as that of the displacement of the object. If the trolley is brought back to the starting point, the object also returns to its initial position. The force of static friction on the object, in this case, acts in the same direction as that of its displacement throughout the journey and hence

$$\oint \boldsymbol{F}_{friction} \cdot d\boldsymbol{r} > 0 \ .$$

Clearly, in either case, the work done by the force of friction on the object around a closed path is not zero.† Friction, thus, is not a conservative force and there is no such thing as frictional potential energy.

S27 *Since we cannot include the effect of friction in the potential energy function, isn't the definition of total energy $E = T + V$ useless in the presence of friction ?*

T27 Not quite. Suppose the total force on a particle is a sum of a conservative force $\mathbf{F}_C$ and a non-conservative force $\mathbf{F}_{NC}$:

$$\boldsymbol{F} = \boldsymbol{F}_C + \boldsymbol{F}_{NC} \ .$$

By work-energy theorem

$$\begin{aligned} \delta T &= \boldsymbol{F} \cdot \delta \boldsymbol{r} \\ &= \boldsymbol{F}_C \cdot \delta \boldsymbol{r} + \boldsymbol{F}_{NC} \cdot \delta \boldsymbol{r} \end{aligned}$$

Now, since a potential energy can be defined for a conservative force, we can write

$$\delta T = - \delta V + \boldsymbol{F}_{NC} \cdot \delta \boldsymbol{r} \ .$$

This, after integration, can be written in the form

$$\Delta E = \Delta (T + V) = \int_1^2 \boldsymbol{F}_{NC} \cdot d\boldsymbol{r} \ . \tag{7}$$

Thus, the work done by the non-conservative forces equals the change in the total energy.

S28 *It will help if you take some illustrative problems at this point.*

T28 Suppose a body starts from rest and slides down a rough inclined plane, the coefficient of kinetic friction between the body and the surface being μ. If the distance

† In the second case, work done on the trolley by the force of static friction between the body and the trolley is negative. The net work done by the static friction on the assembly is zero.

travelled by the body along the plane is ℓ and if the angle the inclined plane makes with the horizontal is 0 (see Fig.6.3), calculate the speed with which the body reaches the ground.

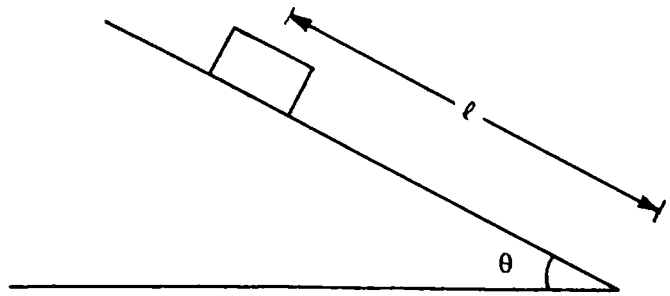

Fig.6.3 A body sliding down a rough inclined plane.

S29 *This doesn't seem to be a difficult problem. The normal force on the body must be equal to the component of the weight of the body perpendicular to the plane. So* $N = mg\cos\theta$.

The force of friction, therefore, is $F_{friction} = \mu\, mg\cos\theta$.

The net force on the body, acting down the inclined plane, is the resultant of the force of friction and the component of the weight down the plane : $mg\sin\theta - \mu\, mg\cos\theta$.

Hence the acceleration of the body $a = g\sin\theta - \mu\, g\cos\theta$.

It is given that the initial velocity $v_i = 0$ *and the distance travelled* $s = \ell$. *The kinematical relation* $v_f^2 - v_i^2 = 2\,a\,s$ *gives the final velocity* $v_f = \sqrt{2g\ell(\sin\theta - \mu\cos\theta)}$.

T29 That is right. But, alternatively, the problem may be solved employing energy considerations. Because of the simultaneous presence of the force of friction and gravity, the form of the work-energy theorem as stated in T27 is appropriate.

The total energy of the body at the highest point is its potential energy :

$E_i = mg\,\ell\sin\theta$.

At the lowest point all the energy of the body is in the form of kinetic energy :

$E_f = \frac{1}{2} mv_f^2$.

Hence $\Delta E = \frac{1}{2} mv_f^2 - mg\ell \sin\theta$.

The force of friction on the body, as you have found, is $F_{friction} = \mu mg \cos\theta$. Therefore, the work done by the force of friction on the body while it slides down the inclined plane is $W = -\mu mg\ell \cos\theta$. Using Eq.(7) we get

$$\frac{1}{2} mv_f^2 - mg\ell \sin\theta = -\mu mg\ell \cos\theta$$

which gives the same final velocity.

In this problem the two methods are equally easy. But this is not always the case. Take another example.

Consider a frictionless loop-the-loop apparatus, with radius R, as shown in Fig.6.4. What is the minimum height H from which the body must be released so that it just completes a loop ?

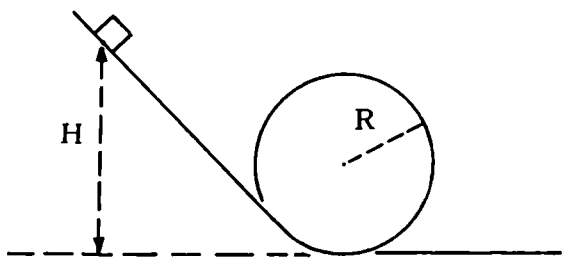

Fig.6.4 A loop-the-loop apparatus.

S30 *Oh, that is a standard problem. At the highest point of the circular track, in general, the weight of the body and the normal force on the body by the track together provide the necessary centripetal force on the body. When the body just clears the highest point, the normal force on it is zero. Therefore*

$$mg = \frac{mv^2}{R} .$$

The total energy of the body at this point, thus, is

$$\frac{1}{2} mv^2 + mg(2R)$$
$$= \frac{5}{2} mgR .$$

If the body starts from rest, from a height H, then it must have so much of potential energy at the initial position. Clearly, the height H = 5R/2.

T30 That is quite correct. Note you did not solve for the detailed motion of the body along the track using Newton's laws. You solved the problem using energy considerations only. This was possible because the condition for clearing the highest point that you employed depends only on the magnitude of the velocity there. This is what happens in many situations. You can get some interesting features from energy considerations alone even if you cannot solve the full dynamics of a problem.

Conservation of Energy

S31 *Eq.(7) seems to imply that the law of conservation of energy is not valid when friction is present. Is that so ?*

T31 You must get this clear. Eq.(6) expresses conservation of total energy (kinetic + potential) of a body moving under an external conservative force. The conservation of total energy of an isolated system, on the other hand, is a general principle of physics. It is important to appreciate the difference between the two. In particular, the word 'total' does not mean the same thing in the two statements. The difference is sometimes made clear by calling E of Eq.(6) as the total mechanical energy.

When friction is present, the total mechanical energy of a body is not conserved. This is what is expressed in Eq.(7). But if you consider the system consisting of the body, the sources of the forces (both conservative and non-conservative) and the environment in interaction with the body, then the total energy of the isolated system thus defined is conserved. Note that the total energy, here, includes not only the

mechanical energy but also the various other possible forms of energy such as internal thermal energy, sound, radiation etc.

For example, the total mechanical energy of a body moving on a rough table is not conserved. However, the body and the table together form an (approximately) isolated system. The total energy of this system, which includes the kinetic energy of the body and the internal (thermal) energy of the body and the table, is (approximately) conserved. We say 'approximately' because we are ignoring, for example, energy lost in the form of sound and other radiations. If everything is included, energy conservation is exact.

S32 *You said that Eq.(6) refers to the conservation of total mechanical energy of a body under an external conservative force. Why did you emphasize the word 'external' here ?*

T32 This is an important point. You see, forces on a body are always exerted by some other bodies. In accordance with Newton's III law, the body exerts equal and opposite forces on the other bodies, and thereby affects their motion. In general, therefore, we must consider the dynamics of the interacting system of bodies as a whole. In that case, you cannot view the motion of any one body as a motion under fixed external agencies of force.

In some situations, however, the agencies i.e. other bodies causing forces on a body may be practically stationary. This may happen if these other bodies are far more massive than the body under consideration, or if they are constrained by some other means. For example, for the problem of motion of a stone under earth's gravity, the earth may be regarded as practically stationary. In such situations it is approximately correct to treat the problem as a single-body problem under a fixed external force. This amounts to a great simplification. Problems of this kind are known in physics as external force problems. Indeed much of our discussion so far implicitly pertains to this situation.

S33 *What is the generalization of the energy conservation equation for a system of interacting particles ?*

T33 For an isolated system of particles interacting via conservative forces between pairs of particles, it can be shown using Newton's laws that the sum of the kinetic energy of

all the particles and the potential energy between pairs of particles is conserved :

$$\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} V_{ij} = \text{constant} .$$

where V_{ij} is the potential energy of mutual interaction between the i^{th} particle and the j^{th} particle. (The sum denoting the potential energy of the system contains pairs of terms such as V_{12} , V_{21} etc. This amounts to double counting. The factor of ½ compensates for this.)

However, when all bodies in a system except one, say the k^{th} body, are practically stationary, the total kinetic energy of the system is just the kinetic energy of the k^{th} body. The summation denoting the total kinetic energy of the system, then, reduces to a single term : $\frac{1}{2}m_k v_k^2$. Similarly, the potential energy, which is determined by all the particles of the system, reduces to a function of $\mathbf{r}_k$ alone.

Thus, when all particles except one, of a system of particles, are constrained from moving, the problem reduces to that of motion of a single particle under external forces; and the total mechanical energy of the particle is conserved.

S34 *It seems to me that energy is not conserved as we go from one frame of reference to another. For example, relative to a moving train's frame of reference, all the surroundings : the trees, buildings, even the distant mountains have a kinetic energy of motion. Relative to the ground's frame these same objects are at rest and possess no kinetic energy. What is the source of the huge extra energy of objects in the moving train's frame of reference ?*

T34 This is a major confusion among students which you must clear up. First, be clear about what the energy conservation law says. It says that <u>in any given inertial frame of reference</u> the total energy of an isolated system is <u>unchanged in time</u>. The law does <u>not</u> claim that the total energy is unchanged from one frame to another in relative motion. That claim would be absurd. Kinetic energy of an object does transform to a different value as we go from one frame to another. This means that the total energy of an isolated system is a constant in time provided that the frame is inertial; but the value of this constant varies from one inertial frame to another.

It is, therefore, meaningless to look for a physical agency or a source which provides the 'huge extra energy' to the mountain in the moving train's frame of reference. That

kind of energy arithmetic is to be done in a given frame of reference, not across different frames of reference. The difference in total energy between frames in relative motion has purely kinematic origin.

Another way of putting the same thing is as follows. For the train to come to a certain uniform velocity, it must be accelerated from rest for a certain time. During this time the train's frame is non-inertial, and all objects in the universe experience pseudo-forces relative to this frame. It is the work done by the pseudo-forces that results in the increased kinetic energy of the objects : the trees, the buildings, the mountains and so on. Of course, the pseudo-forces have no material origin and thus the increase in the kinetic energy, as stated above, has purely kinematic origin.

S35 *One last query about the law of conservation of energy. Is it an independent principle of physics, or does it follow from the laws of motion ?*

T35 As already said, for a mechanical system of particles interacting via conservative forces the principle follows from Newton's laws of motion.

For any system of charged particles and electromagnetic fields, the principle can be seen as a consequence of the laws of electromagnetism and Newton's laws. In this case the total energy comprises of the kinetic energy of the particles and the energy associated with the electric and magnetic fields.

In general, however, the principle is accepted as an independent law of physics (The first law of thermodynamics).

There is another related principle : we believe that under identical conditions an experiment performed today will yield the same results as it did at some time in the past, or as it will yield if performed any time in the future. That is, the laws of physics possess time-displacement invariance. In more advanced theory, the principle of conservation of energy can be shown to be a consequence of this invariance. Of course, the validity of the time-displacement invariance cannot be proved from any first principles; it is an assumption corroborated by our experience.

Conservation of Momentum

S36 *How about the conservation of linear momentum ? Does it follow from the laws of motion ? or is it also an independent principle of physics ?*

T36 You must never forget to add the qualifier, 'of an isolated system'. The total linear momentum of an isolated system is conserved.

In ordinary mechanics of a system of particles, this principle follows straight from the II and the III law of motion.

We have seen that, in the relativistic domain, Newton's II law is applicable with a properly modified definition of momentum. Also, Newton's III law is valid in the relativistic domain for contact forces. Hence, for a system of particles interacting via contact forces, the total linear momentum is conserved in this domain.

Electromagnetic forces are non-contact forces, and in the relativistic description, they are seen as mediated by fields which propagate at the speed of light. The III law is then no longer true as, for example, for forces between moving charges. It turns out that it is possible to define momentum of an electromagnetic field so that the total momentum of a system of charged particles and fields is conserved. The proof of this statement is beyond our scope here.

In general, however, the principle of momentum conservation must also be accepted as an independent principle of physics.

I said in T35 that the principle of energy conservation is related to the time-displacement invariance. Likewise, the momentum conservation principle is related to another invariance principle called the space-translation invariance. This means that under identical conditions, the outcome of an experiment will be the same whether it is performed here, there or anywhere else. This amounts to the physically reasonable assumption that space is homogeneous.

S37 *Is the momentum conservation law true in a non-inertial frame of reference ?*

T37 Of course not. In a non-inertial frame, an isolated system i.e. a system without any external material force is still subject to pseudo-forces. Its momentum, therefore, is

not conserved. You must apply this law only in an inertial frame.

S38 *For both elastic and inelastic collisions the total linear momentum is conserved: but total kinetic energy is conserved for elastic collisions only. What is behind this ?*

T38 During collision between bodies A and B, whatever the nature of forces between them, the force on A by B is the negative of the force on B by A at each instant according to the III law. This is enough to ensure that the total momentum of the two bodies is conserved.

There is no analogous reason that guarantees the conservation of total kinetic energy of the two bodies. No doubt, the total energy of an isolated system is conserved. But as we know, this total energy includes energy in any form : kinetic energy, translational and rotational, potential energy associated with any conservative forces in the system, internal (thermal) energy of bodies etc. In an inelastic collision, depending on the nature of the interaction between the two bodies, part of the initial kinetic energy may get transformed into any of the other possible forms. Consequently the total kinetic energy of bodies need not be conserved in a collision, although their total momentum is always conserved.

S39 *Then what is it that ensures the conservation of total kinetic energy in case of elastic collisions ?*

T39 It is all a question of the nature of the force that acts during a collision. If this force happens to be conservative, the collision is elastic. For example, if during collision the bodies simply get compressed, say like elastic springs, then the force between them is conservative like the usual spring force. This, of course, is not a realistic picture of the force between colliding bodies, but it illustrates the point.

S40 *I am still not clear about the connection between elastic collisions and the conservative nature of force. Please elaborate.*

T40 We have already seen (T19) that, for a conservative force, a potential energy can be defined such that the sum of kinetic energy and potential energy of the system is conserved. Now, the force associated with collision has one obvious property : it is *non-zero only* for such separation between the bodies that they can be in contact.

Thus, for the collision of two billiard balls, the force is zero when the separation between them (i.e. the distance between their centres) exceeds the sum of their radii. In this entire region the potential energy is a constant, which we can choose to be zero (See T22).

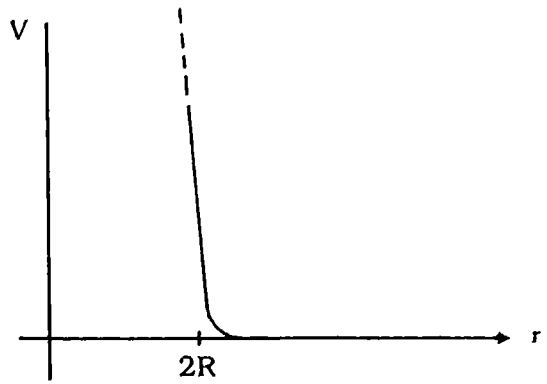

Fig.6.5 A schematic graph of potential energy of interaction between two balls, each of radius R, as a function of separation between their centres.

Fig.(6.5) shows a possible graph for potential energy versus separation, corresponding to a collision of two balls each of radius R. Note V(r) and the force between the balls both are zero for r > 2R and shoot up for r < 2R, showing the sharply increasing repulsive force as the balls get compressed further and further. [Actually, the graph is sensible only for values of r near 2R. For smaller r, the bodies are compressed beyond their elastic limit; and the notion of potential energy is inapplicable.]

Thus for a conservative force which is zero at large distances, the initial total energy is simply the initial total kinetic energy. Likewise, the final total energy equals the final total kinetic energy. Since the sum of kinetic and potential energy is a constant throughout for any conservative force , the total initial kinetic energy of the system must equal the total final kinetic energy.

S41 *What happens during the short time of collision when the bodies are in contact ? Is the total momentum conserved during this time ? Is the total kinetic energy conserved during this time, if the collision is elastic ?*

T41 That should be clear to you from what we have already discussed. At each instant during the collision, whatever be the type of collision (elastic or inelastic), the III law

is always valid. Since the mutual forces are equal and opposite, the change in momentum of body A is the negative of the change in momentum of body B. The total momentum of bodies A and B, therefore, remains the same at every instant before, during and after the collision.

If the collision is elastic, the initial (i.e. when the bodies are apart before collision) total kinetic energy is equal to the final (i.e. when the bodies go apart again after collision) total kinetic energy. During the collision, however, the total kinetic energy is not conserved even if the collision is elastic. The kinetic energy decreases initially during an elastic collision, being progressively transformed into potential energy. The bodies may come to a rest momentarily if the collision is one-dimensional. During the later part of the collision, the potential energy gets transformed back to kinetic energy. The sum of the total kinetic energy and the potential energy is, of course, constant at every instant for an elastic collision.

S42 *Are the laws of conservation of momentum and energy independent of each other ? I mean, in a process, if one condition is satisfied, will the other be satisfied automatically, or is that a separate requirement ?*

T42 These are two independent conservation laws. One does not follow from the other. They impose separate requirements on a process. For a process to occur, the conditions of energy and momentum conservation must be simultaneously satisfied. This greatly restricts possible processes in nature.

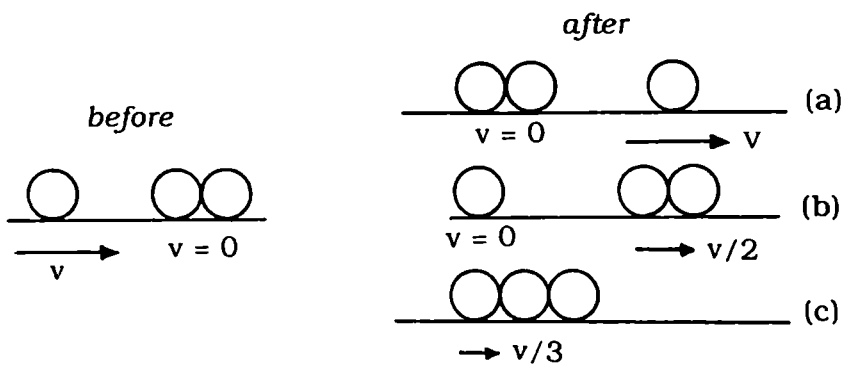

Fig.6.6 Outcomes of a one-dimensional collision. The only outcome allowed by energy-momentum conservation is the one in (a).

Fig.(6.6) gives three outcomes of a one-dimensional collision. All of them satisfy the condition of momentum conservation. Yet if we are given that the collision is elastic, only the process in Fig.(6.6 a) is possible since that alone conserves total kinetic energy. The remaining two processes are not possible outcomes of an elastic collision.

S43 *Any more examples of the application of energy-momentum conservation ?*

T43 This is something that one keeps at the back of one's mind all the time. Indeed, in atomic and sub-atomic domain, energy-momentum conservation is one of the first things to check to rule out a process. For example, consider a process in which a free proton gets converted into a neutron by emitting a pion :

$$p \rightarrow n + \pi^+.$$

It is forbidden by energy-momentum conservation, although it satisfies many other requirements such as charge conservation, baryon number conservation[†] etc.

S44 *How do you see that ?*

T44 Suppose the proton is initially at rest. Conservation of momentum, then, requires

$$\boldsymbol{p}_n = - \boldsymbol{p}_{\pi^+} ;$$

and conservation of total energy requires

$$m_p c^2 = \sqrt{p_n^2 c^2 + m_n^2 c^4} + \sqrt{p_{\pi^+}^2 c^2 + m_{\pi^+}^2 c^4} .$$

It is clearly impossible to satisfy the above equation since $m_p < m_n$.

S45 *But what if I give enough kinetic energy to the proton so that its total energy is greater than the sum of the rest-mass energy of n and π^+ ? Relativity allows energy-mass*

---

† Particles capable of experiencing the nuclear strong force are called 'hadrons'. They are classified in two catagories : hadrons with integral spin, such as pions, are called mesons; and hadrons with half odd integral spin, such as protons and neutrons, are called baryons. Each particle has an associated baryon number B. Mesons have B = 0, baryons have B = 1 and antibaryons have B = -1. Baryon number, like charge, is an additive quantum number. Total baryon number is conserved in every process of nature observed so far.

conversion, so a sufficiently energetic proton should be able to decay into a neutron and a pi-meson.

T45 That is a serious error. You made this mistake because you are looking at energy conservation in isolation from momentum conservation. As already stressed, a process must satisfy both the conservation laws at the same time.

If initially the proton has momentum $\mathbf{p}_p$ then energy-momentum conservation conditions are:

$$\boldsymbol{p}_p = \boldsymbol{p}_n + \boldsymbol{p}_{\pi^+} ;$$

$$\sqrt{p_p^2c^2 + m_p^2c^4} = \sqrt{p_n^2c^2 + m_n^2c^4} + \sqrt{p_{\pi^+}^2c^2 + m_{\pi^+}^2c^4}$$

It can be shown algebraically that the two conditions are impossible to be satisfied simultaneously. That is, the process is forbidden.

One could arrive at the same conclusion by a more physical argument. Let us consider the situation relative to a frame that is moving with the same velocity as the proton. In this frame, the proton is at rest. By the same reasoning as in T44, the process is not allowed relative to this frame. Now, the same process cannot be forbidden relative to one frame and allowed relative to another. Thus, the decay of a high energy proton into a neutron and a π^+ meson is not possible.

S46 *The decay $A \rightarrow B + C$ is not allowed if $m_A < m_B + m_C$. But the reaction $A + B \rightarrow C + D$ is allowed even if $m_A + m_B < m_C + m_D$? Why ?*

T46 It is all a question of being able to satisfy simultaneously both energy and momentum conservation. We have seen this is not possible in the first case. But for the second case it is possible, provided the reactants are given enough energy.

There is another important thing to note in case of two-body decay : $A \rightarrow B + C$. For the decay of particle A at rest, the decay products must always come out with definite energy and momentum of definite magnitude, as fixed by equations like those in T44.

S47 *Yes, I see that. But what is so significant about it ?*

T47 Suppose in the decay of a certain particle at rest, seemingly, there are only two products. However, rather than having definite energies, the two decay products are experimentally observed to come out with a spectrum of energies. This experimental finding is incompatible with a two-body decay, and indicates that we are missing to detect some other particle(s).

This is what happened historically with β decay. The β decay of a nucleus apparently gave only two products : the daughter nucleus and an electron. But the electrons did not have a fixed energy; their energy spectrum was observed to be continuous. Pauli reconciled the finding with the energy-momentum conservation by suggesting that, in β decay, the parent nucleus changes to the daughter nucleus by emitting, not just an electron, but an electron and some other particle.† The existence of this particle is now established beyond doubt. It is called antineutrino.

S48 *Let us return to a classical problem. Consider the motion of a ball on a frictionless floor. If it hits a wall, it rebounces with some speed. The wall is stationary throughout, and hence the change in its momentum is zero. The momentum of the ball has changed sign. Clearly, the total momentum of the isolated (wall + ball) system has changed. How can that be ?*

T48 First of all be clear about the isolated system. The wall is rigidly (almost) connected to the ground, so that by 'wall' we should mean really the whole earth. Thus what we are talking of is the isolated ball + earth system.††

There is no violation of the law of momentum conservation here. Remember this law is so universally true that in any argument showing its apparent violation, one must detect some error. The error in your argument is in thinking that the wall remains strictly stationary. Let us see this better :

During collision the wall exerts a force on the ball. By the III law, the ball exerts an equal and opposite force on the wall. This causes the wall to recoil back. However, since the wall is enormously more massive than the ball, its recoil velocity is negligible compared to the ball. That is what makes you think it is at rest.

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† Some additional arguments were also advanced for this hypothesis which need not concern us here.

†† Strictly, this system is not isolated; it is moving under the sun's gravity. Also the earth's frame is not strictly inertial. However, for the problem at hand these effects are negligible.

S49 *Qualitatively this seems alright; but can we not see it more precisely ?*

T49 That is quite straightforward to see. Let us denote the mass of the ball by m and that of the wall by M. Let the velocity of the ball before collision be v_b, and let the velocities of the ball and the wall after collision be v_b' and v_w' respectively. Momentum conservation requires

$$m v_b = m v_b' + M v_w' ,$$

and energy conservation, assuming the collision to be elastic, gives

$$\frac{1}{2} m v_b^2 = \frac{1}{2} m v_b'^2 + \frac{1}{2} M v_w'^2 .$$

These equations can be easily solved to obtain velocities after collision as

$$v_b' = \frac{m - M}{m + M} v_b ; \qquad v_w' = \frac{2m}{m + M} v_b .$$

Evidently, as $M/m \to \infty$, $v_w' \to 0$ and $v_b' \to - v_b$; i.e. for M very large compared to m, v_w' is negligible compared to v_b, and the ball bounces back with the same speed. The momenta of the two, after collision, are given by

$$p_b' = \frac{m - M}{m + M} p_b ; \qquad p_w' = \frac{2M}{m + M} p_b .$$

Notice, although $v_w' \to 0$ as $M/m \to \infty$, $p_w' \to 2p_b$ in the same limit. I.e. the momentum of the wall is <u>not</u> negligible compared to the momentum of the ball (in fact, it is twice as much here). It is the velocity that is negligible. (The negligible velocity times the very large mass becomes a non-negligible momentum.)

S50 *For the elastic collision that one often considers, the ball is taken to rebounce with the same speed as the initial speed. If, as you say, the wall recoils with some momentum, it should carry away some kinetic energy. The ball must, therefore, rebounce with reduced energy. Is that not so ?*

T50 You should appreciate an important point : when a massive body recoils, it may carry non-negligible momentum and yet its recoil energy can be negligible. In the collision between the ball and the wall, the wall shares momentum but not energy.

S51 *How is that possible ?*

T51 From the expressions for v_w' above, we can write the expression for the kinetic energy of the wall, T_w', after collision :

$$T_w' = \frac{4mM}{(m + M)^2} T_b ,$$

where T_b is the kinetic energy of the ball before collision. It is clear that $T_w' \to 0$ as $M/m \to \infty$.

A simpler way of seeing this is to express kinetic energy in terms of momentum :

$$T_b' = \frac{p_b'^2}{2m} : \qquad T_w' = \frac{p_w'^2}{2M} .$$

Clearly, since p_b' and p_w' are comparable, $T_w' << T_b'$ if $M >> m$.

S52 *This seems a rather interesting point. Do you come across such situations elsewhere in physics ?*

T52 Yes, at many places. For example, when an atom in an energy state E_1 decays to another energy state E_2, a photon of frequency ν is emitted. Do you know how ν is related to the energies E_1 and E_2 ?

S53 *I know that. It is given in the Bohr model by $E_1 - E_2 = h\nu$.*

T53 You are almost right, but you have ignored the small recoil effect. When an atom at rest decays, the momentum of the emitted photon $h\nu/c$ must be equal and opposite to the recoil momentum of the atom. So the recoil energy of the atom is given by

$$\frac{p^2}{2m} = \frac{h^2\nu^2}{2mc^2} ,$$

and hence the photon energy is given by the equation

$$E_1 - E_2 = h\nu + \frac{h^2\nu^2}{2mc^2} .$$

The recoil energy is negligible as compared to the photon energy since hν is of the order of a few ev whereas mc^2 for an atom is of the order of Bev.

Similar considerations are relevant when a nucleus decays from an excited level E_1 to a lower level E_2, emitting a gamma ray. In this case the energy of the emitted photon is much larger (of the order of Mev) and, therefore, the recoil momentum and energy of the nucleus much greater. There is a way, however, of preventing the nucleus from recoiling. If the nucleus is embedded in a crystal, then, roughly speaking, the entire crystal recoils. Because of its large effective mass, the recoiling system carries little energy, as in the ball + wall problem, while it carries away as much momentum as required by momentum conservation. This is the idea underlying a well-known method of high-resolution nuclear spectroscopy : Mossabauer spectroscopy.

S54 *Let me return to our discussion in T43. According to that a process like* $p \rightarrow n + e^+ + \nu$ *is not allowed by energy-momentum conservation since proton mass is less than the neutron mass. Yet this is the process that gives rise to* β^+ *emission. How do you understand that ?*

T54 You must be careful here. Energy-momentum conservation says that a free proton cannot decay as above. If the proton is bound, the decay is possible.

S55 *I did not get you. How does the proton being 'bound' help ?*

T55 In this case, the decay to be considered is:

(p + rest of the parent nucleus) → (n + rest of the daughter nucleus) + e^+ + ν.

The important thing is that the rest of the nucleus is not isolated from the process; it shares energy and momentum as appropriate for conservation. The only condition is that the mass of the parent nucleus should be greater than the sum of the masses of the products. The binding energy of the proton with the rest of the parent nucleus is not the same as the binding energy of the neutron with the rest of the daughter nucleus. Indeed you can say that the difference in these binding energies is what provides the necessary extra energy that the proton needs in order to decay to a more massive neutron.

S56 *This looks like an important point. Any other example ?*

T56 We come across a related, but not completely similar, situation in photoelectric emission. A process

$$e^-_{free} + \gamma \rightarrow e^-$$

is not allowed by energy-momentum conservation. [To see this, observe the process relative to the final electron's rest frame. The initial electron and the photon must have equal and opposite momenta in this frame. Consequently, energy conservation is violated in the process.]

S57 *But photoelectric emission is due to the absorption of a photon from incident light by an electron in a metal. How does it take place at all if the process above is not allowed ?*

T57 The answer is that the initial electron is not free; it is bound in the metal. The following process is not forbidden by energy-momentum conservation :

$$\text{lattice} + e^-_{bound} + \gamma \rightarrow \text{lattice} + e^-_{free}.$$

At this point you must note one thing. Though the lattice shares momentum in the process, its share of energy is completely negligible. The reason this is so is exactly as in the ball + wall problem. The mass of the lattice is so large that $p^2_{lattice}/2M$ is nearly zero.

S58 *One final question. A nuclear reaction is always cited as an example of mass-energy interconversion. Why should this be so only for a nuclear reaction. Why can't mass be converted into energy or vice-versa in a chemical reaction ?*

T58 This is a common misconception. Actually, as far as mass-energy interconversion is concerned, there is no difference <u>in principle</u> between a nuclear reaction and a chemical reaction: the difference is only in the magnitudes of the energies involved. Mass gets converted into energy (or vice-versa) even in a chemical reaction.

S59 *I am surprised. Please elaborate.*

T59 Let us see how the energetics of a chemical reaction was looked at before the advent of the ideas of mass-energy interconversion. Take, for example, the reaction :

$NaOH + HCl \rightarrow NaCl + H_2O$.

This is what was believed :

1) A chemical reaction is simply a rearrangement of atoms in different combinations. No atoms are created or destroyed in a chemical reaction. (This is the reason we need to balance a chemical equation.)

2) Atoms are bound with different binding energies in different chemical compounds. The total binding energy in the reactants may be different from the total binding energy in the products. This difference in total binding energies on the two sides of a reaction appears as heat produced or absorbed. In an exothermic reaction as above, the binding energy on the right ($B.E._{r.h.s.}$) is greater than the binding energy on the left ($B.E._{l.h.s.}$). Thus,

$B.E._{r.h.s.} - B.E._{l.h.s.} = Q$, the net heat produced.

3) Mass of a compound is a sum of the masses of its constituent atoms; e.g. mass of NaCl = mass of Na + mass of Cl; and similarly for other compounds. Since there is no creation or destruction of individual atoms, the total mass of the reactants is equal to the total mass of the products. This is the principle of conservation of mass in a chemical reaction.

S60 *What is wrong with this view-point ?*

T60 The first two statements continue to be correct. Statement (3), however, is not strictly correct from a relativistic view-point.

The basic thing you must know is that, in relativistic mechanics, the rest mass of a bound composite system is not equal to the sum of the rest masses of its constituent parts. By definition, binding energy (B.E.) is the energy required to unbound the parts of a system. By mass-energy equivalence, this implies for a bound system such as NaCl

$m_{NaCl}c^2 + B.E. = m_{Na}c^2 + m_{Cl}c^2$,

where m_x is the mass of x. This means

$m_{NaCl} = m_{Na} + m_{Cl} - \delta m$; $\delta m = B.E./c^2$.

δm is called 'mass defect'.

Now, since the total binding energy is not the same on the two sides of the reaction, it is clear from the above equation that the total mass defects on the two sides are unequal. This implies, in turn, that the total mass on the left side (reactants) is not the same as the total mass of the right side (products). Thus mass is not conserved in a chemical reaction.

In short, from a relativistic view-point, Statement (2) above (T59) is completely equivalent to the statement that heat produced in a chemical reaction is due to the conversion of part of the mass of the reactants into energy.

S61 *If that is so, why does everybody cite nuclear reactions as example of mass-energy interconversion ?*

T61 It is simply a matter of magnitudes. In chemical compounds the binding energies are of the order of ev. The nucleonic binding energy in nuclei, on the other hand, is typically a few Mev. Consequently, the mass defects in nuclei are much larger than those in chemical compounds. In fact the mass defect of a chemical compound is hardly measurable. In nuclei mass defects are appreciable. Therefore mass-energy interconversion is usually invoked in nuclear reactions and not in chemical reactions.